భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad
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Matrix Theory

22 Eigenvalues and Eigenvectors III
(Symmetric matrices and the spectral theorem)

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The eigenvectors are orthogonal. This is no accident.

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Let $A\mathbf{u} = \lambda\mathbf{u}$, $A\mathbf{v} = \mu\mathbf{v}$, $\lambda \neq \mu$. Then

$$\lambda(\mathbf{u} \cdot \mathbf{v}) = (A\mathbf{u})^T\mathbf{v} = \mathbf{u}^T A^T\mathbf{v} = \mathbf{u}^T A\mathbf{v} = \mu(\mathbf{u} \cdot \mathbf{v}).$$

So $(\lambda - \mu)(\mathbf{u} \cdot \mathbf{v}) = 0$, and since $\lambda \neq \mu$ we get $\mathbf{u} \cdot \mathbf{v} = 0$.

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Symmetry was used in exactly one place: $A^T = A$.

The spectral theorem

Spectral Theorem for symmetric matrices

Let A be a real symmetric $n \times n$ matrix. Then:

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2. eigenspaces for distinct eigenvalues are mutually orthogonal;
3. the geometric multiplicity equals the algebraic multiplicity for every eigenvalue, so A **is** diagonalizable;
4. in fact $A = QDQ^T$ with Q **orthogonal** ($Q^TQ = I$, so $Q^{-1} = Q^T$).

The spectral theorem

Why this matters

No defective cases, no complex arithmetic, and the inverse of Q is free. Symmetric matrices are where the theory is cleanest — which is fortunate, since covariance matrices, Gram matrices, Hessians, and graph Laplacians are all symmetric.

Orthogonal diagonalization in practice

For $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, normalize the eigenvectors:

$$\mathbf{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \mathbf{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

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Procedure

1. Find each eigenspace.
2. Apply Gram–Schmidt *within* each eigenspace of dimension ≥ 2 (across eigenspaces orthogonality is automatic).
3. Normalize every vector.

Reading $A = QDQ^T$ geometrically

Expand the factorization:

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \cdots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T .$$

$$Ax = \sum_{i=1}^{n} \lambda_i q_i q_i^T x$$

$\frac{a a^T}{a^T a} \cdot b$

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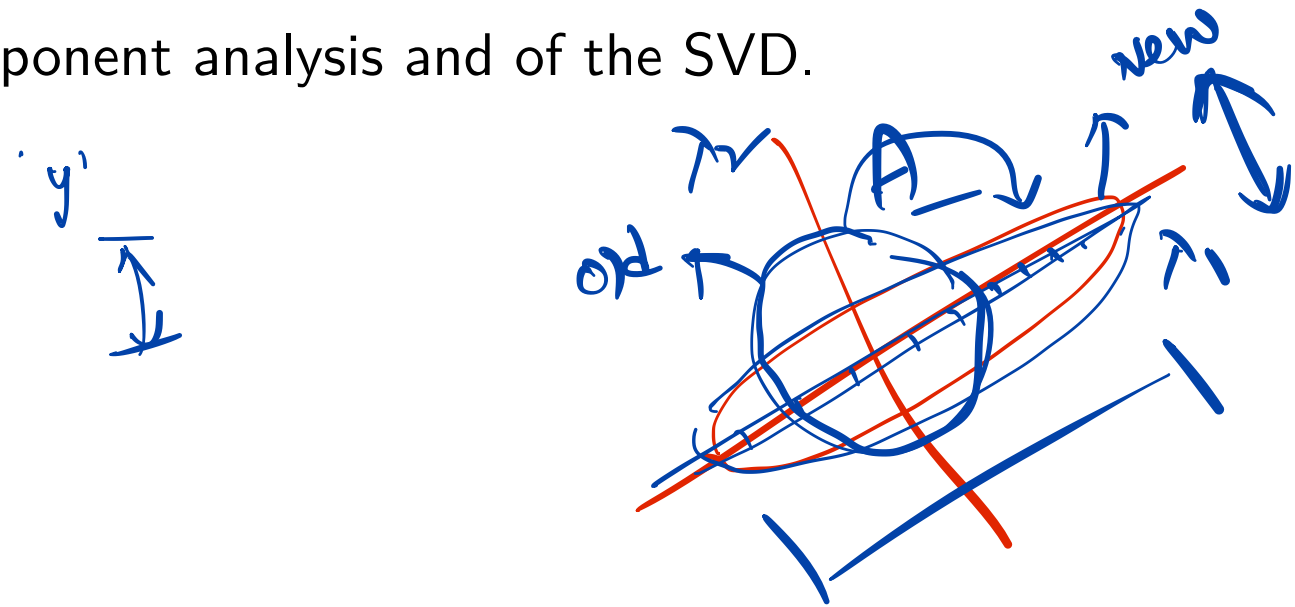
A symmetric matrix decomposes space into perpendicular axes and stretches each by its own factor.

Reading $A = QDQ^T$ geometrically

Applying A to the unit circle gives an ellipse with axes along the $\mathbf{q}_i$ and semi-axis lengths $|\lambda_i|$. For our example: the unit circle becomes an ellipse stretched by 4 along $(1, 1)$ and by 2 along $(1, -1)$.

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Applying A to the unit circle gives an ellipse with axes along the $\mathbf{q}_i$ and semi-axis lengths $|\lambda_i|$. For our example: the unit circle becomes an ellipse stretched by 4 along $(1, 1)$ and by 2 along $(1, -1)$. This is the germ of principal component analysis and of the SVD.



Iterating a matrix

Many models have the form

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$$\boxed{\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n}$$

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The central idea of this lecture: The eigenvectors are the *modes* of the system; the eigenvalues are their *growth rates*. Long-run behaviour is decided by which $|\lambda_i|$ is largest.

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Stability is entirely a statement about $\max_i |\lambda_i|$ — the *spectral radius*.

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Factor out the largest:

$$\mathbf{x}_k = \lambda_1^k \left[c_1 \mathbf{v}_1 + c_2 \left(\frac{\lambda_2}{\lambda_1} \right)^k \mathbf{v}_2 + \cdots \right].$$

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Interpretation

Whatever you start with, the system aligns itself with the dominant eigenvector. The *shape* stabilizes even when the *size* grows.

Example: a population model

Two age classes, juveniles j_k and adults a_k:

$$\begin{bmatrix} j_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1.5 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} j_k \\ a_k \end{bmatrix}.$$

$j_{k+1} = 1.5\, a_k$

$a_{k+1} = 0.6\, j_k + 0.8\, a_k$

Next year

Current year

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$|\lambda_1| > 1$: the population **grows about 43% per period**, and the age distribution converges to the direction of $\mathbf{v}_1$ regardless of the starting mix.

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Two cities, with yearly migration:

$$P = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}, \quad \mathbf{x}_{k+1} = P\mathbf{x}_k.$$

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The other eigenvalue is $\operatorname{tr} - 1 = 0.92$, with $|0.92| < 1$. So $\mathbf{x}_k \to \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$:

in the long run 37.5% live in city 1 — **whatever the initial distribution**.

Summary slide

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$AP = PD$

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One idea: find the coordinates in which the problem is diagonal.

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- Assuming the eigenvalues of $A + B$ are sums of eigenvalues, or that A and B share eigenvectors.
- Forgetting that eigenvectors are determined only up to a nonzero scalar — your answer and the textbook's may differ legitimately.

Exercises

1. Find the eigenvalues and eigenspaces of $\begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix}$ and verify the eigenvectors are orthogonal. Explain why that was predictable.
2. Show that $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is not diagonalizable.
3. Diagonalize $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$ and use it to compute A^{10}.
4. Prove: if λ is an eigenvalue of an invertible A, then λ^{-1} is an eigenvalue of A^{-1}, with the same eigenvector.
5. Find the steady-state vector of $\begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix}$ and determine how fast convergence occurs.
6. Give a 3×3 matrix with a single eigenvalue of algebraic multiplicity 3 and geometric multiplicity 2.
7. For which c is $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ diagonalizable?

Rough Work

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