



Matrix Theory

21 Eigenvalues and Eigenvectors II (Similarity and diagonalization)

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The motivating computation



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Write $\mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2$. Then

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No matrix multiplication at all — just two scalar powers. Let us make that into a matrix identity.



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Similar matrices



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Proof

$$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P, \text{ so}$$

$$\det(B - \lambda I) = \det(P^{-1}) \det(A - \lambda I) \det(P) = \det(A - \lambda I),$$

$$\text{since } \det(P^{-1}) \det(P) = 1.$$



Similarity is a change of coordinates: same linear map, different basis.

The diagonalization theorem



A is *diagonalizable* if $A = PDP^{-1}$ for some diagonal D and invertible P .

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Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable $\iff A$ has n linearly independent eigenvectors.

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Equivalently

A is diagonalizable $\iff \mathbb{R}^n$ has a basis consisting of eigenvectors of A (an *eigenbasis*).

Why the theorem is true



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Write $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ and $D = \text{diag}(\lambda_1, \dots, \lambda_n)$.

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$$AP = PD \iff A\mathbf{v}_i = \lambda_i\mathbf{v}_i \text{ for every } i.$$

And $AP = PD$ can be rewritten as $A = PDP^{-1}$ exactly when P is invertible, i.e. exactly when the $\mathbf{v}_i$ are independent.

Diagonalizing: a full example

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$



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$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}. \text{ Step 1. } p(\lambda) = -(\lambda - 1)(\lambda + 2)^2, \text{ so}$$

$$\lambda = 1, -2, -2.$$

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$$\text{reduces to the single equation } x_1 + x_2 + x_3 = 0, \text{ giving } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

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$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \text{ Step 4. Three independent eigenvectors in } \mathbb{R}^3:$$

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Two remarks on that example



- P is **not unique**. Rescale a column, reorder the columns (reordering D to match), or pick a different basis for E_{-2} – all give valid factorizations.

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Here, column 2: $A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 + 3 \\ 3 - 5 \\ -3 + 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \checkmark$

A matrix that cannot be diagonalized



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The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

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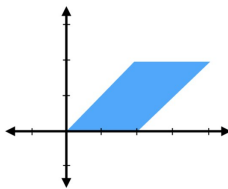
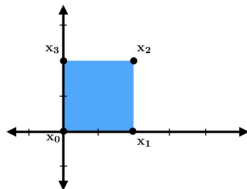
The eigenspace is only **one**-dimensional. There is no second independent eigenvector, so no eigenbasis of $\mathbb{R}^2$, so A is *not diagonalizable*.

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Geometrically: the shear fixes one (horizontal) axis and tilts everything else. There is exactly one invariant line.

Shear

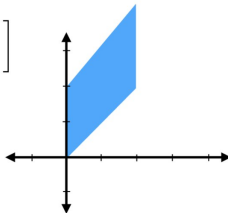
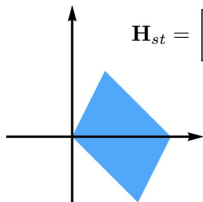


Shear in x:

$$\mathbf{H}_{xs} = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$

Arbitrary shear:

$$\mathbf{H}_{st} = \begin{bmatrix} 1 & s \\ t & 1 \end{bmatrix}$$



Shear in y:

$$\mathbf{H}_{ys} = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$$

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Definition

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Theorem(Diagonalizability criterion)

A (over $\mathbb{C}$, or over $\mathbb{R}$ when p splits) is diagonalizable $\iff$ for **every** eigenvalue the geometric and algebraic multiplicities agree.

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A matrix failing this is called *defective*.

Comparing our examples



Matrix	λ	alg. mult.	geom. mult.
$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$	5	1	1
	3	1	1
$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$	1	1	1
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Only the last row has a mismatch — and only the last matrix fails to diagonalize.

A sufficient condition worth remembering



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Theorem

If an $n \times n$ matrix has n **distinct** eigenvalues, it is diagonalizable.

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If an $n \times n$ matrix has n **distinct** eigenvalues, it is diagonalizable.

Proof

Each eigenvalue contributes an eigenvector; distinct eigenvalues give independent eigenvectors (Lecture MT-20). That is n independent vectors in $\mathbb{R}^n$, hence a basis.

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The converse is false

I has the single eigenvalue 1 and is already diagonal. Repeated eigenvalues are a *warning*, not a verdict — you must check the eigenspace dimension.

Powers made easy



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and inductively

$$\boxed{A^k = PD^kP^{-1}}, \quad D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}.$$

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Raising a diagonal matrix to a power is entrywise. The whole cost of A^k collapses to n scalar powers plus two matrix products. The same identity works for $A^{-1} = PD^{-1}P^{-1}$ (when no $\lambda_i = 0$).

Where real eigenvalues run out



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The quarter-turn rotation

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The fix

Work in $\mathbb{C}^n$. Everything proved so far — eigenspaces, characteristic polynomial, diagonalization — carries over verbatim with complex scalars, and now p always has n roots.

Complex eigenvectors

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$



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Row 2 gives $x_1 = ix_2$, so $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (and row 1 is $-i$ times row 2 — always a useful check).

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Theorem

If A is **real** and $A\mathbf{v} = \lambda\mathbf{v}$, then $A\bar{\mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}}$.

So complex eigenvalues of real matrices come in conjugate pairs; here

$\lambda = -i$ has eigenvector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ for free.

What a complex eigenvalue means



Rotation–scaling

Let A be a real 2×2 matrix with eigenvalue $\lambda = a - bi$ ($b \neq 0$) and eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1}, \quad P = [\operatorname{Re} \mathbf{v} \quad \operatorname{Im} \mathbf{v}], \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

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And C factors as

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r=|\lambda|} \underbrace{\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}}_{\text{rotate by } \varphi}.$$

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Reading

A complex eigenvalue $\lambda = re^{i\varphi}$ means: in suitable coordinates, A **rotates by φ and scales by $r = |\lambda|$** . The modulus controls growth; the argument controls rotation speed.



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- $A^k = PD^kP^{-1}$ turns matrix powers into scalar powers.
- Complex eigenvalues $re^{i\varphi}$ encode rotation by φ and scaling by r .

Rough Work



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