భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad
॥ नवोन्मेषेण देश निर्माणम् ॥

Matrix Theory

21 Eigenvalues and Eigenvectors II (Similarity and diagonalization)

Dr. Konda Reddy Mopuri
Dept. of AI, IIT Hyderabad
July-Nov 2026

The motivating computation

Suppose you need A^{20} for $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$.

The motivating computation

Suppose you need A^{20} for $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Twenty matrix multiplications?

The motivating computation

Suppose you need A^{20} for $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Twenty matrix multiplications?

Instead, recall

$$A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The motivating computation

Suppose you need A^{20} for $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Twenty matrix multiplications?

Instead, recall

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A^{20}\mathbf{x} = c_1 5^{20}\mathbf{v}_1 + c_2 3^{20}\mathbf{v}_2.$$

The motivating computation

Suppose you need A^{20} for $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Twenty matrix multiplications?

Instead, recall

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Write $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then

$$A^{20}\mathbf{x} = c_1 5^{20}\mathbf{v}_1 + c_2 3^{20}\mathbf{v}_2.$$

No matrix multiplication at all — just two scalar powers. Let us make that into a matrix identity.

Similar matrices

A and B are *similar* if $B = P^{-1}AP$ for some invertible P.

Similar matrices

A and B are *similar* if $B = P^{-1}AP$ for some invertible P.

Similar matrices have the same characteristic polynomial, hence the same eigenvalues with the same algebraic multiplicities.

Similar matrices

A and B are *similar* if $B = P^{-1}AP$ for some invertible P.

Similar matrices have the same characteristic polynomial, hence the same eigenvalues with the same algebraic multiplicities.

Proof

$B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(A - \lambda I)P$, so

$$\det(B - \lambda I) = \det(P^{-1})\det(A - \lambda I)\det(P) = \det(A - \lambda I),$$

since $\det(P^{-1})\det(P) = 1$.

Similar matrices

Similarity is a change of coordinates: same linear map, different basis.

The diagonalization theorem

A is *diagonalizable* if $A = PDP^{-1}$ for some diagonal D and invertible P.

The diagonalization theorem

A is *diagonalizable* if $A = PDP^{-1}$ for some diagonal D and invertible P.

Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable $\iff$ A has n linearly independent eigenvectors.

In that case $A = PDP^{-1}$ where the columns of P are those eigenvectors and the diagonal entries of D are the corresponding eigenvalues, **in the same order**.

The diagonalization theorem

A is *diagonalizable* if $A = PDP^{-1}$ for some diagonal D and invertible P.

Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable $\iff$ A has n linearly independent eigenvectors.

In that case $A = PDP^{-1}$ where the columns of P are those eigenvectors and the diagonal entries of D are the corresponding eigenvalues, **in the same order**.

Equivalently

A is diagonalizable $\iff$ $\mathbb{R}^n$ has a basis consisting of eigenvectors of A (an *eigenbasis*).

Why the theorem is true

Write $P = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$.

Why the theorem is true

Write $P = [\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_n\,]$ and $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Column by column:

$$AP = [\,A\mathbf{v}_1 \;\cdots\; A\mathbf{v}_n\,], \qquad PD = [\,\lambda_1\mathbf{v}_1 \;\cdots\; \lambda_n\mathbf{v}_n\,].$$

Why the theorem is true

Write $P = [\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_n\,]$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Column by column:

$$AP = [\,A\mathbf{v}_1 \;\cdots\; A\mathbf{v}_n\,], \qquad PD = [\,\lambda_1\mathbf{v}_1 \;\cdots\; \lambda_n\mathbf{v}_n\,].$$

So

$$AP = PD \iff A\mathbf{v}_i = \lambda_i\mathbf{v}_i \text{ for every } i.$$

Why the theorem is true

Write $P = [\,\mathbf{v}_1 \;\cdots\; \mathbf{v}_n\,]$ and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Column by column:

$$AP = [\,A\mathbf{v}_1 \;\cdots\; A\mathbf{v}_n\,], \qquad PD = [\,\lambda_1\mathbf{v}_1 \;\cdots\; \lambda_n\mathbf{v}_n\,].$$

So

$$AP = PD \iff A\mathbf{v}_i = \lambda_i\mathbf{v}_i \text{ for every } i.$$

And $AP = PD$ can be rewritten as $A = PDP^{-1}$ exactly when P is invertible, i.e. exactly when the $\mathbf{v}_i$ are independent.

Diagonalizing: a full example

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

Diagonalizing: a full example

$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. **Step 1.** $p(\lambda) = -(\lambda - 1)(\lambda + 2)^2$, so

$\lambda = 1, -2, -2.$

Diagonalizing: a full example

$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. **Step 1.** $p(\lambda) = -(\lambda - 1)(\lambda + 2)^2$, so

$\lambda = 1, -2, -2$. **Step 2.** $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Diagonalizing: a full example

$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. **Step 1.** $p(\lambda) = -(\lambda - 1)(\lambda + 2)^2$, so

$\lambda = 1, -2, -2$. **Step 2.** $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. **Step 3.** $\lambda = -2$: $A + 2I$

reduces to the single equation $x_1 + x_2 + x_3 = 0$, giving $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$,

$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Diagonalizing: a full example

$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$. **Step 1.** $p(\lambda) = -(\lambda - 1)(\lambda + 2)^2$, so

$\lambda = 1, -2, -2$. **Step 2.** $\lambda = 1$: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. **Step 3.** $\lambda = -2$: $A + 2I$

reduces to the single equation $x_1 + x_2 + x_3 = 0$, giving $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$,

$\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. **Step 4.** Three independent eigenvectors in $\mathbb{R}^3$:

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Two remarks on that example

- P is **not unique**. Rescale a column, reorder the columns (reordering D to match), or pick a different basis for E_{-2} – all give valid factorizations.

Two remarks on that example

- P is **not unique**. Rescale a column, reorder the columns (reordering D to match), or pick a different basis for E_{-2} – all give valid factorizations.
- The order must be **consistent**: column i of P pairs with entry i of D.

Two remarks on that example

- P is **not unique**. Rescale a column, reorder the columns (reordering D to match), or pick a different basis for E_{-2} – all give valid factorizations.
- The order must be **consistent**: column i of P pairs with entry i of D.
- **Always check:** verify $AP = PD$ rather than computing P^{-1}. It is faster and catches ordering mistakes.

Two remarks on that example

- P is **not unique**. Rescale a column, reorder the columns (reordering D to match), or pick a different basis for E_{-2} – all give valid factorizations.
- The order must be **consistent**: column i of P pairs with entry i of D.
- **Always check:** verify $AP = PD$ rather than computing P^{-1}. It is faster and catches ordering mistakes.

Here, column 2: $A \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1+3 \\ 3-5 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$. ✓

A matrix that cannot be diagonalized

The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

A matrix that cannot be diagonalized

The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Triangular, so $p(\lambda) = (1-\lambda)^2$: the only eigenvalue is $\lambda = 1$, algebraic multiplicity 2.

A matrix that cannot be diagonalized

The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Triangular, so $p(\lambda) = (1-\lambda)^2$: the only eigenvalue is $\lambda = 1$, algebraic multiplicity 2.

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad x_2 = 0, \quad E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

A matrix that cannot be diagonalized

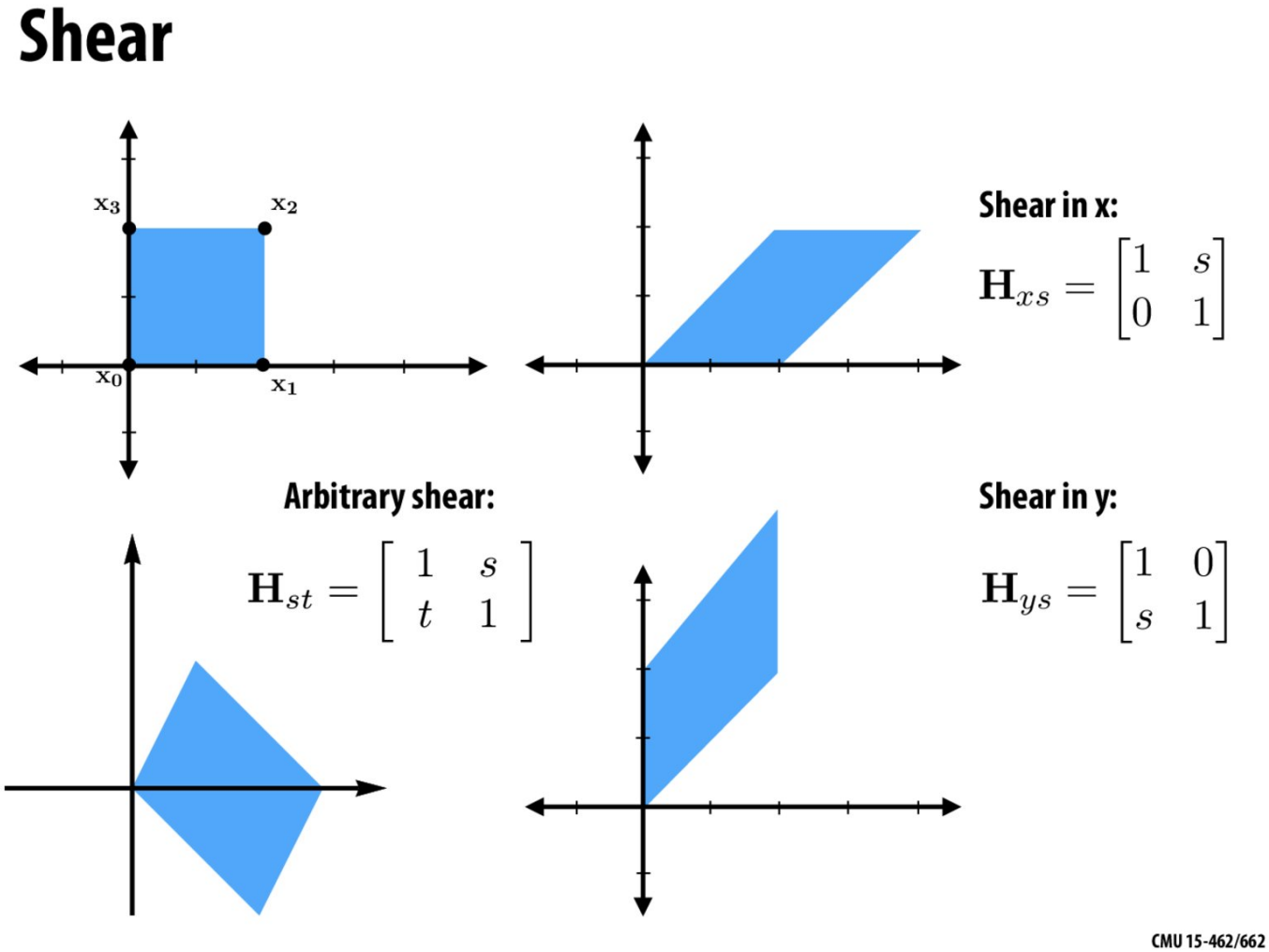
The shear $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Triangular, so $p(\lambda) = (1-\lambda)^2$: the only eigenvalue is $\lambda = 1$, algebraic multiplicity 2.

$$A - I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Longrightarrow \quad x_2 = 0, \quad E_1 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

The eigenspace is only **one**-dimensional. There is no second independent eigenvector, so no eigenbasis of $\mathbb{R}^2$, so A is *not diagonalizable*.

A matrix that cannot be diagonalized

Geometrically: the shear fixes one (horizontal) axis and tilts everything else. There is exactly one invariant line.



Credits: 15462 CS.CMU

Geometric multiplicity

Definition

The *geometric multiplicity* of λ is $\dim E_\lambda = \dim \mathrm{Nul}(A - \lambda I)$.

Geometric multiplicity

Definition

The *geometric multiplicity* of λ is $\dim E_\lambda = \dim \mathrm{Nul}(A - \lambda I)$.

Theorem

For every eigenvalue,

$$1 \leq \text{ geometric multiplicity } \leq \text{ algebraic multiplicity.}$$

Geometric multiplicity

Definition

The *geometric multiplicity* of λ is $\dim E_\lambda = \dim \mathrm{Nul}(A - \lambda I)$.

Theorem

For every eigenvalue,

$$1 \leq \text{ geometric multiplicity } \leq \text{ algebraic multiplicity.}$$

Theorem(Diagonalizability criterion)

A (over $\mathbb{C}$, or over $\mathbb{R}$ when p splits) is diagonalizable $\iff$ for **every** eigenvalue the geometric and algebraic multiplicities agree.

Geometric multiplicity

Definition

The *geometric multiplicity* of λ is $\dim E_\lambda = \dim \mathrm{Nul}(A - \lambda I)$.

Theorem

For every eigenvalue,

$$1 \leq \text{ geometric multiplicity } \leq \text{ algebraic multiplicity.}$$

Theorem(Diagonalizability criterion)

A (over $\mathbb{C}$, or over $\mathbb{R}$ when p splits) is diagonalizable $\iff$ for **every** eigenvalue the geometric and algebraic multiplicities agree.

A matrix failing this is called *defective*.

Comparing our examples

Matrix	λ	alg. mult.	geom. mult.
$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$	5	1	1
	3	1	1
$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$	1	1	1
	-2	2	2
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	1	2	$\mathbf{1}$

Comparing our examples

Matrix	λ	alg. mult.	geom. mult.
$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$	5	1	1
	3	1	1
$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$	1	1	1
	-2	2	2
$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	1	2	$\mathbf{1}$

Only the last row has a mismatch — and only the last matrix fails to diagonalize.

A sufficient condition worth remembering

Theorem

If an $n \times n$ matrix has n **distinct** eigenvalues, it is diagonalizable.

A sufficient condition worth remembering

Theorem

If an $n \times n$ matrix has n **distinct** eigenvalues, it is diagonalizable.

Proof

Each eigenvalue contributes an eigenvector; distinct eigenvalues give independent eigenvectors (Lecture MT-20). That is n independent vectors in $\mathbb{R}^n$, hence a basis.

A sufficient condition worth remembering

Theorem

If an $n \times n$ matrix has n **distinct** eigenvalues, it is diagonalizable.

Proof

Each eigenvalue contributes an eigenvector; distinct eigenvalues give independent eigenvectors (Lecture MT-20). That is n independent vectors in $\mathbb{R}^n$, hence a basis.

The converse is false

I has the single eigenvalue 1 and is already diagonal. Repeated eigenvalues are a *warning*, not a verdict — you must check the eigenspace dimension.

Powers made easy

If $A = PDP^{-1}$ then

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1},$$

Powers made easy

If $A = PDP^{-1}$ then

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1},$$

and inductively

$$\boxed{A^k = PD^kP^{-1}}, \qquad D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}.$$

Powers made easy

If $A = PDP^{-1}$ then

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1},$$

and inductively

$$\boxed{A^k = PD^kP^{-1}}, \qquad D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}.$$

Raising a diagonal matrix to a power is entrywise. The whole cost of A^k collapses to n scalar powers plus two matrix products.

Powers made easy

If $A = PDP^{-1}$ then

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1},$$

and inductively

$$\boxed{A^k = PD^kP^{-1}}, \qquad D^k = \begin{bmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k \end{bmatrix}.$$

Raising a diagonal matrix to a power is entrywise. The whole cost of A^k collapses to n scalar powers plus two matrix products. The same identity works for $A^{-1} = PD^{-1}P^{-1}$ (when no $\lambda_i = 0$).

Where real eigenvalues run out

The quarter-turn rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Where real eigenvalues run out

The quarter-turn rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$p(\lambda) = \lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i.$$

Where real eigenvalues run out

The quarter-turn rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$p(\lambda) = \lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i.$$

No real eigenvalues: a rotation by 90° leaves *no* line invariant. The algebra is telling us something true about the geometry.

Where real eigenvalues run out

The quarter-turn rotation

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$p(\lambda) = \lambda^2 + 1 = 0 \quad \Longrightarrow \quad \lambda = \pm i.$$

No real eigenvalues: a rotation by 90° leaves *no* line invariant. The algebra is telling us something true about the geometry.

The fix

Work in $\mathbb{C}^n$. Everything proved so far — eigenspaces, characteristic polynomial, diagonalization — carries over verbatim with complex scalars, and now p always has n roots.

Complex eigenvectors

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

Complex eigenvectors

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

Row 2 gives $x_1 = ix_2$, so $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (and row 1 is $-i$ times row 2 — always a useful check).

Complex eigenvectors

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

Row 2 gives $x_1 = ix_2$, so $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (and row 1 is $-i$ times row 2 — always a useful check). Check: $A \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$. ✓

Complex eigenvectors

For $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\lambda = i$:

$$A - iI = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}.$$

Row 2 gives $x_1 = ix_2$, so $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ (and row 1 is $-i$ times row 2 — always a useful check). Check: $A \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ i \end{bmatrix} = i \begin{bmatrix} i \\ 1 \end{bmatrix}$. ✓

Theorem

If A is **real** and $A\mathbf{v} = \lambda\mathbf{v}$, then $A\overline{\mathbf{v}} = \overline{\lambda}\,\overline{\mathbf{v}}$.

So complex eigenvalues of real matrices come in conjugate pairs; here $\lambda = -i$ has eigenvector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$ for free.

What a complex eigenvalue means

Rotation–scaling

Let A be a real 2×2 matrix with eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1}, \qquad P = [\operatorname{Re}\mathbf{v} \quad \operatorname{Im}\mathbf{v}], \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

What a complex eigenvalue means

Rotation–scaling

Let A be a real 2×2 matrix with eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1}, \qquad P = [\operatorname{Re}\mathbf{v} \;\; \operatorname{Im}\mathbf{v}], \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

And C factors as

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r=|\lambda|} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotate by } \varphi}.$$

What a complex eigenvalue means

Rotation–scaling

Let A be a real 2×2 matrix with eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and eigenvector $\mathbf{v}$. Then

$$A = PCP^{-1}, \qquad P = [\operatorname{Re}\mathbf{v} \;\; \operatorname{Im}\mathbf{v}], \qquad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

And C factors as

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r=|\lambda|} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotate by } \varphi}.$$

What a complex eigenvalue means

And C factors as

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r=|\lambda|} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotate by } \varphi}.$$

What a complex eigenvalue means

And C factors as

$$C = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scale by } r=|\lambda|} \underbrace{\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}}_{\text{rotate by } \varphi}.$$

Reading

A complex eigenvalue $\lambda = re^{i\varphi}$ means: in suitable coordinates, A **rotates by φ and scales by $r = |\lambda|$.** The modulus controls growth; the argument controls rotation speed.

Summary

- $A = PDP^{-1}$: columns of P are eigenvectors, diagonal of D the eigenvalues, matched in order.

Summary

- $A = PDP^{-1}$: columns of P are eigenvectors, diagonal of D the eigenvalues, matched in order.
- Diagonalizable $\iff$ an eigenbasis exists $\iff$ geometric multiplicity $=$ algebraic multiplicity for every λ.

Summary

- $A = PDP^{-1}$: columns of P are eigenvectors, diagonal of D the eigenvalues, matched in order.
- Diagonalizable $\iff$ an eigenbasis exists $\iff$ geometric multiplicity $=$ algebraic multiplicity for every λ.
- n distinct eigenvalues suffices but is not necessary.

Summary

- $A = PDP^{-1}$: columns of P are eigenvectors, diagonal of D the eigenvalues, matched in order.
- Diagonalizable $\iff$ an eigenbasis exists $\iff$ geometric multiplicity $=$ algebraic multiplicity for every λ.
- n distinct eigenvalues suffices but is not necessary.
- $A^k = PD^kP^{-1}$ turns matrix powers into scalar powers.

Summary

- $A = PDP^{-1}$: columns of P are eigenvectors, diagonal of D the eigenvalues, matched in order.
- Diagonalizable $\iff$ an eigenbasis exists $\iff$ geometric multiplicity $=$ algebraic multiplicity for every λ.
- n distinct eigenvalues suffices but is not necessary.
- $A^k = PD^kP^{-1}$ turns matrix powers into scalar powers.
- Complex eigenvalues $re^{i\varphi}$ encode rotation by φ and scaling by r.

Rough Work

Rough Work