



Matrix Theory

20 Eigenvalues and Eigenvectors - I

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Eigenvectors



భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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- So far we looked at $Ax = b$

Eigenvectors



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- So far we looked at $Ax = b$
- We solve $Ax = \lambda x$ by simplifying

Eigenvectors



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- So far we looked at $Ax = b$
- We solve $Ax = \lambda x$ by simplifying
- Determinants give this transition

$$x = \vec{v}$$

Eigenvectors



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- So far we looked at $Ax = b$
- We solve $Ax = \lambda x$ by simplifying
- Determinants give this transition
- All our matrices are square now

A matrix usually turns a vector



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Let

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

A matrix usually turns a vector



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Let

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

Apply it to a few vectors:

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix},$$

Diagram: Red arrows point from the first column of the input vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to the first and second rows of the output vector $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$

Diagram: Red arcs are drawn under the second column of the input vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the first and second rows of the output vector $\begin{bmatrix} 6 \\ 2 \end{bmatrix}$.

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

Diagram: Red arcs are drawn under the entire input vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and the entire output vector $\begin{bmatrix} 8 \\ 12 \end{bmatrix}$.

A matrix usually turns a vector



Let

$$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}.$$

Apply it to a few vectors:

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \end{bmatrix}.$$

In each case the output points in a *new* direction.

A matrix usually turns a vector



Let

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Apply it to a few vectors:

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In each case the output points in a *new* direction.

The question for the next three classes

Are there directions that A does *not* turn?

Two special directions



For the same $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

(Handwritten red checkmark above the equation)

$$A \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

(Handwritten red arrows pointing to the vectors and scalars, with 'i/p' and 'o/p' written below them)

Two special directions



For the same $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$:

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

These vectors are only *stretched* (and, in the second case, *flipped*). The matrix acts on them like a single number.

$$\begin{aligned} \underline{A} \underline{v}_1 &= \lambda \underline{v}_1 \\ \underline{A} \underline{v}_2 &= \mu \underline{v}_2 \end{aligned}$$

Two special directions



For the same $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$: $N(A - 7I)$

$N(A + 4I)$

$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4 \begin{bmatrix} 6 \\ -5 \end{bmatrix}.$$

These vectors are only *stretched* (and, in the second case, flipped). The matrix acts on them like a single number.

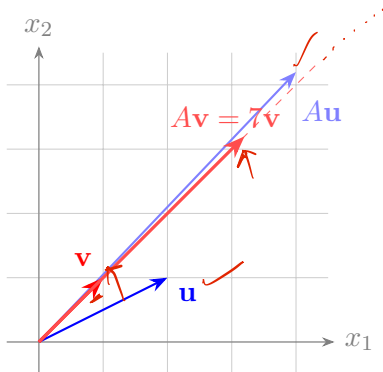
Why we care

On such a direction, the complicated object A collapses to scalar multiplication. If we can decompose a problem into these directions, linear algebra becomes arithmetic.

Example Visualization



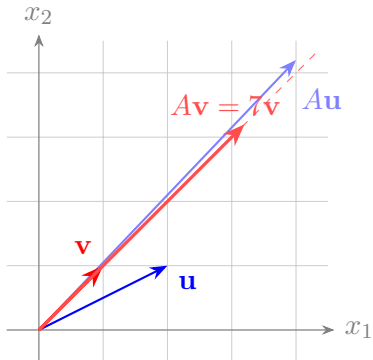
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Example Visualization



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$\mathbf{u}$ leaves its line; $\mathbf{v}$ stays on its line. The dashed line is *invariant* under A .

Definition



Eigenvector and eigenvalue

Let A be an $n \times n$ matrix. A **nonzero** vector $\mathbf{v} \in \mathbb{R}^n$ is an *eigenvector* of A if

$$A\mathbf{v} = \lambda\mathbf{v}$$

for some scalar λ . That scalar λ is the *eigenvalue* of A corresponding to $\mathbf{v}$.

$$A \cdot \mathbf{0} = \lambda \mathbf{0}$$

$$A\mathbf{v} = \lambda\mathbf{v}$$

$$A\mathbf{v} = \mathbf{0}$$

Definition



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Three points to note:

- A must be **square**. Otherwise $A\mathbf{v}$ and $\mathbf{v}$ live in different spaces.

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- $\mathbf{v} \neq \mathbf{0}$ is part of the definition. ($A\mathbf{0} = \lambda\mathbf{0}$ always holds and says nothing.)

Definition



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- A must be **square**. Otherwise $A\mathbf{v}$ and $\mathbf{v}$ live in different spaces.
- $\mathbf{v} \neq \mathbf{0}$ is part of the definition. ($A\mathbf{0} = \lambda\mathbf{0}$ always holds and says nothing.)
- $\lambda = 0$ is allowed. $A\mathbf{v} = \mathbf{0}$ with $\mathbf{v} \neq \mathbf{0}$ means 0 is an eigenvalue.

Verifying a vector



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Question

Is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

$$\underline{Av = \lambda v}$$

Verifying a vector



Question

Is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

$$A \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 - 12 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}.$$

✓ λv

$$- \frac{9}{3} = -3$$

Verifying a vector



Question

Is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

$$A \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 - 12 \\ 15 - 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix}.$$

Is $\begin{bmatrix} -9 \\ 11 \end{bmatrix}$ a multiple of $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$? It would need $-9 = 3c$ and $11 = -2c$, i.e. $c = -3$ and $c = -5.5$. **No.**

Verifying a vector



Question

Is $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ an eigenvector of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

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Note the asymmetry

Verifying a given vector is a one-line computation. *Finding* the eigenvectors is the real work.

Verifying a value



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Question

Is $\lambda = 7$ an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

$$AV = \lambda V$$

Verifying a value



Question

Is $\lambda = 7$ an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

We need a nonzero $\mathbf{v}$ with $A\mathbf{v} = 7\mathbf{v}$, i.e.

$$\underline{A\mathbf{v} - 7\mathbf{v} = \mathbf{0}} \iff \underline{(A - 7I)\mathbf{v} = \mathbf{0}}.$$

Verifying a value



Question

Is $\lambda = 7$ an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

We need a nonzero $\mathbf{v}$ with $A\mathbf{v} = 7\mathbf{v}$, i.e.

$$A\mathbf{v} - 7\mathbf{v} = \mathbf{0} \iff (A - 7I)\mathbf{v} = \mathbf{0}.$$

Careful: $A - 7$ is meaningless; you must write $A - 7I$.

Verifying a value



Question

Is $\lambda = 7$ an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$? $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda = \lambda \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

We need a nonzero $\mathbf{v}$ with $A\mathbf{v} = 7\mathbf{v}$, i.e.

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Careful: $A - 7$ is meaningless; you must write $A - 7I$.

$$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

$$\lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \checkmark$$

Verifying a value



Question

Is $\lambda = 7$ an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$?

We need a nonzero $\mathbf{v}$ with $A\mathbf{v} = 7\mathbf{v}$, i.e.

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$$A - 7I = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \xrightarrow{\text{row reduce}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

A free variable exists, so there are nonzero solutions: $\mathbf{v} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Yes, 7 is an eigenvalue.

For an eigenvalue λ of A , the *eigenspace* of A for λ is

$$E_\lambda = \text{Nul}(A - \lambda I) = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\}.$$

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- E_λ is a **subspace**: it is a null space, so it is closed under addition and scalar multiplication and contains $\mathbf{0}$.
- $E_\lambda \setminus \{\mathbf{0}\}$ is exactly the set of eigenvectors for λ .
- λ is an eigenvalue $\iff \text{Nul}(A - \lambda I) \neq \{\mathbf{0}\} \iff A - \lambda I$ is **not invertible**.

$$\begin{array}{l} A\mathbf{v} = \lambda\mathbf{v} \\ \hline (A - \lambda I)\mathbf{v} = \mathbf{0} \end{array}$$

$$\{\mathbf{0}\}$$

Eigenspace example



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Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and take $\lambda = 2$.

Eigenspace example



Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and take $\lambda = 2$.

$$(A - 2I) = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Handwritten row reduction steps:

$$\begin{array}{ccc} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

Handwritten equation:

$$N(A - 2I) = x_2 \begin{bmatrix} x_1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} x_3$$

Handwritten constraint equation:

$$2x_1 - x_2 + 6x_3 = 0$$

Eigenspace example



Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and take $\lambda = 2$.

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Eigenspace example



Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and take $\lambda = 2$.

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$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

E_{λ_i}

So $E_2 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$, a **plane** in $\mathbb{R}^3$.

Eigenspace example



Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ and take $\lambda = 2$.

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$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

So $E_2 = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$, a **plane** in $\mathbb{R}^3$.

Every vector in that plane is simply doubled by A .

Finding eigenvalues: Characteristic Equation



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So far we tested a λ someone handed us. To *find* them:

Finding eigenvalues: Characteristic Equation



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So far we tested a λ someone handed us. To *find* them:

λ is an eigenvalue $\iff (A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution

$$A\mathbf{v} = \lambda\mathbf{v}$$

Finding eigenvalues: Characteristic Equation



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$$\begin{aligned}\lambda \text{ is an eigenvalue} &\iff (A - \lambda I)\mathbf{x} = \mathbf{0} \text{ has a nontrivial solution} \\ &\iff A - \lambda I \text{ is singular}\end{aligned}$$

Finding eigenvalues: Characteristic Equation



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λ is an eigenvalue $\iff \underline{(A - \lambda I)\mathbf{x} = \mathbf{0}}$ has a nontrivial solution

$\iff \underline{A - \lambda I}$ is singular

$\iff \underline{\det(A - \lambda I) = 0}.$

Finding eigenvalues: Characteristic Equation



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$\iff A - \lambda I$ is singular

$\iff \det(A - \lambda I) = 0$.

Characteristic equation

$$\det(A - \lambda I) = 0$$

is the *characteristic equation* of A ; the left side is the *characteristic polynomial* $p(\lambda)$.

The characteristic polynomial



Theorem

If A is $n \times n$, then $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n in λ , with leading term $(-1)^n \lambda^n$.

$$\prod_{i=1}^n (a_{ii} - \lambda)$$



The characteristic polynomial



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Consequences, over $\mathbb{C}$:

- A has at least one and at most n distinct eigenvalues.

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Consequences, over $\mathbb{C}$:

- A has at least one and at most n distinct eigenvalues.
- Counted with repetition, there are exactly n .

The characteristic polynomial



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$$\underline{Ax = \lambda x}$$

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Consequences, over $\mathbb{C}$:

- A has at least one and at most n distinct eigenvalues. }
- Counted with repetition, there are exactly n .
- Over $\mathbb{R}$ there may be **none**.



Algebraic multiplicity

The algebraic multiplicity of λ_i is its multiplicity as a root of $p(\lambda)$.

2×2 case

$$\text{For } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$



2×2 case



$$\text{For } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right) = \underline{(a - \lambda)(d - \lambda) - bc}.$$

2×2 case



For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\det(A - \lambda I) = \det \left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \right) = (a - \lambda)(d - \lambda) - bc.$$

Expanding:

$$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) = \lambda^2 - \text{tr}(A)\lambda + \det(A).$$

2×2 case



For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

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Worth memorizing

For 2×2 matrices: $\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.$

2×2 example



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$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}. \quad \text{tr } A = 8, \det A = 7 + 8 = 15.$$

2×2 example



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$$\text{Check: } A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 - 4 \\ -4 - 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \quad \checkmark$$

3×3 Example

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}. \text{ Expanding } \det(A - \lambda I) \text{ gives}$$

$$p(\lambda) = -(\lambda - 2)^2(\lambda - 9).$$



$$0 = \begin{vmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{vmatrix}$$

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$$N(A - \lambda I)$$

dim 0

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For $\lambda = 9$:

$$A - 9I = \begin{bmatrix} -5 & -1 & 6 \\ 2 & -8 & 6 \\ 2 & -1 & -1 \end{bmatrix} \longrightarrow \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

$$\text{Check: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \end{bmatrix}. \checkmark$$

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Note on the computation required



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- Expanding $\det(A - \lambda I)$ costs $\approx n!$ operations

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- Expanding $\det(A - \lambda I)$ costs $\approx n!$ operations
- Polynomial root-finding can be cumbersome and numerically unstable
- In practice, softwares use iterative methods (e.g., QR algorithm) that avoid characteristic polynomial

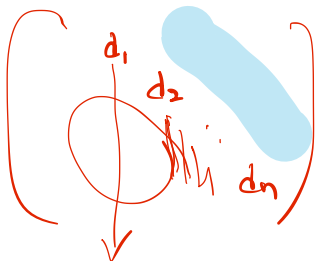
Triangular Matrices give the eigenvalues away



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theorem

If A is triangular, its eigenvalues are the entries on its main diagonal.



Triangular Matrices give the eigenvalues away



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Reason

$A - \lambda I$ is triangular, and the determinant of a triangular matrix is the product of its diagonal entries, so $p(\lambda) = \prod_i (a_{ii} - \lambda)$.

Triangular Matrices give the eigenvalues away



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Example

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{has eigenvalues } 3, 0, 2.$$

Since 0 is an eigenvalue, A is not invertible.

$$(A - \lambda I) = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \quad A \checkmark$$

Zero as an eigenvalue



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A is invertible $\iff 0$ is **not** an eigenvalue of A .

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This attaches eigenvalues to the Invertible Matrix Theorem: add to the list of equivalent statements

- $\det A \neq 0$;
- 0 is not an eigenvalue of A ;
- $\lambda_1 \lambda_2 \cdots \lambda_n \neq 0$.

Invertible Matrix Theorem



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Eigenvalues of related matrices



Suppose $A\mathbf{v} = \lambda\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$. Then:

- $A^k\mathbf{v} = \lambda^k\mathbf{v}$ (apply A repeatedly)

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- $(A + cI)\mathbf{v} = (\lambda + c)\mathbf{v}$ (a shift shifts the spectrum)

$$\begin{array}{l} \textcircled{1} \lambda_1 \\ \textcircled{3} \textcircled{4} \\ \hline \lambda_1 \\ \lambda_2 \end{array} \quad \begin{array}{l} \left[\begin{array}{cc} \textcircled{3} & 2 \\ 3 & \textcircled{6} \end{array} \right] \\ \hline \lambda_1 + 2 \\ \lambda_2 + 2 \end{array}$$

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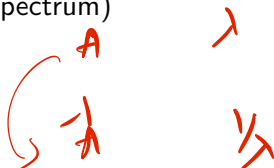
- $A^k\mathbf{v} = \lambda^k\mathbf{v}$ (apply A repeatedly)
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- if A is invertible, $A^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$



$$A A^{-1} = \underline{A A^{-1}} = \underline{I} \underline{v}$$

$$A^{-1} A v = v$$

$$A^{-1} v = \frac{1}{\lambda} v$$

Eigenvalues of related matrices



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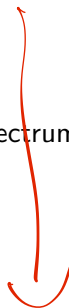
$$A^k \mathbf{v} - \lambda^k \mathbf{v} = 0$$

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In every case the *eigenvector* is unchanged

One fixed direction, many matrices built from A , all acting as scalars on it.

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Not true in general: the eigenvalues of $A + B$ are *not* the sums of those of A and B .

Distinct eigenvalues $\rightarrow$ independent eigenvectors



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If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors of A corresponding to **distinct** eigenvalues $\lambda_1, \dots, \lambda_r$, then $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

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Proof sketch (contradiction)

Let p be least with $\mathbf{v}_{p+1} \in \text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$, say

$$\mathbf{v}_{p+1} = c_1 \mathbf{v}_1 + \dots + c_p \mathbf{v}_p. \quad (*)$$

Apply A : $\lambda_{p+1} \mathbf{v}_{p+1} = c_1 \lambda_1 \mathbf{v}_1 + \dots + c_p \lambda_p \mathbf{v}_p$.

Multiply $(*)$ by λ_{p+1} and subtract:

$$\mathbf{0} = c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \dots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p.$$

By minimality of p the $\mathbf{v}_i$ here are independent, so every $c_i(\lambda_i - \lambda_{p+1}) = 0$. The eigenvalues are distinct, so all $c_i = 0$, forcing $\mathbf{v}_{p+1} = \mathbf{0}$ — impossible.

Summary



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- $A\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$: directions A only scales.

Summary



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