



Matrix Theory

19 Determinants - II

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Dept. of AI, IIT Hyderabad
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Last class



భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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- Geometric interpretation of Determinant (signed area/volume)

However



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- Why is the formula for the determinant $(ad - bc)$, natural?

However



- Why is the formula for the determinant $(ad - bc)$, natural?
- It is in the elimination process!

$$\left. \begin{aligned} ax + by &= l \\ cx + dy &= m \end{aligned} \right\}$$

$$-\frac{d}{b} [ax + by = l]$$

$$\left(c - \frac{ad}{b}\right)x = m - \frac{ld}{b}$$

$$x = \frac{bm - ld}{bc - ad}$$

$$\begin{aligned} -\frac{c}{a} (ax + by = l) \\ cx + dy = m \end{aligned}$$

$$\left(d - \frac{bc}{a}\right)y = m - \frac{lc}{a}$$

$$y = \frac{am - lc}{ad - bc}$$

$$\begin{array}{r} \swarrow \left(\begin{array}{cc|c} a & b & l \\ c & d & m \end{array} \right) \\ \hline \left(\begin{array}{cc|c} a & b & \\ c & d & \end{array} \right) \end{array}$$

History behind the Determinant



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- It was not invented as an abstract algebraic object

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- 17th century noticed this number appearing, unavoidably, in the denominator

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- Determinant came from the Latin determinare, to determine (Gauss used it in 1800s)
- Cauchy (1815) the first comprehensive, organized treatment of determinants as a theory
- Jacobi (1840s) systematized the theory further and connected determinants to calculus

Determinant



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Historical order: solve equations $\rightarrow$ notice a denominator $\rightarrow$ study that denominator $\rightarrow$ discover its properties $\rightarrow$ realize it measures volume

Teaching order: volume $\rightarrow$ axioms $\rightarrow$ formula $\rightarrow$ solve equations ✓

Let's consider the 3×3 case



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- let's define the determinant from the three axioms

Let's consider the 3×3 case



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 - A1 (Normalization): $\det(I) = 1$

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 - A2 (Alternating): swapping any two columns flips the sign

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$$\det(c \cdot A) = c^n \det(A)$$

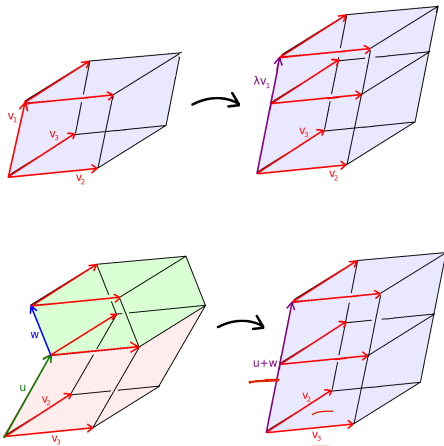
$n \times n$

- let's define the determinant from the three axioms
 - A1 (Normalization): $\det(I) = 1$
 - A2 (Alternating): swapping any two columns flips the sign
 - A3 (Multilinear): $\det$ is linear in each column separately

$$\det(A+B) \neq \det(A) + \det(B) \quad \checkmark$$

$$\begin{aligned} |a_1 \quad \widetilde{a_i + a_j} \quad a_n| &= |a_1 \quad \widetilde{a_i} \quad a_n| + |a_1 \quad \widetilde{a_j} \quad a_n| \\ |a_1 \quad \widetilde{c a_j} \quad a_n| &= c |a_1 \quad \widetilde{a_j} \quad a_n| \end{aligned}$$

Multilinearity



volumes are equal by Cavalieri's principle
(imagine each stack as a stack of thin slices)

Credits: Nick Alger

Let's consider the 3×3 case

- $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$



Let's consider the 3×3 case



- $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$
- Expand each column in the standard basis

$$a_1 = a_{11}e_1 + a_{21}e_2 + a_{31}e_3$$

$$a_2 = a_{12}e_1 + a_{22}e_2 + a_{32}e_3$$

$$a_3 = a_{13}e_1 + a_{23}e_2 + a_{33}e_3$$

Let's consider the 3×3 case



- $A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$
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- By multilinearity (pull sum out of each column slot)
 $\det(A) = \sum_i \sum_j \sum_k a_{i1}a_{j2}a_{k3} \det(\begin{bmatrix} e_i & e_j & e_k \end{bmatrix})$

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- $\det(e_i, e_j, e_k) = 0$ when indices match $\rightarrow$ kills many of these 27 terms

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- $\det(e_i, e_j, e_k) = 0$ when indices match $\rightarrow$ kills many of these 27 terms
- Only 6 survive (permutations of 1, 2, 3)
- $\det(e_i, e_j, e_k) = \text{sign}(\sigma)$ where σ is the permutation (i, j, k)

Determinant of a 3×3 matrix



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$$\det(A) = a_{11}a_{22}a_{33} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} - a_{11}a_{32}a_{23} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23}$$

Determinant of a $n \times n$ matrix



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Leibniz formula :

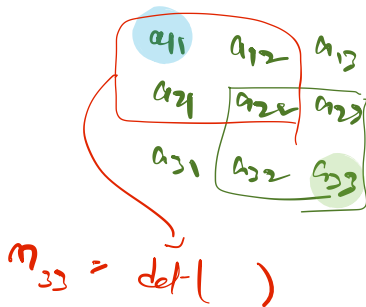
$$\det(A) = \sum_{\sigma \in S_n} \text{sign}(\sigma) \cdot a_{\sigma(1),1} \cdot a_{\sigma(2),2} \cdots a_{\sigma(n),n}$$

Cofactor expansion (Laplace's Theorem)



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- Minor M_{ij} : Determinant of $(n - 1) \times (n - 1)$ submatrix removing row i and column j



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- Minor M_{ij} : Determinant of $(n - 1) \times (n - 1)$ submatrix removing row i and column j
- Cofactor $C_{ij} = (-1)^{i+j} \cdot M_{ij}$

Cofactor expansion (Laplace's Theorem)



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- Minor M_{ij} : Determinant of $(n - 1) \times (n - 1)$ submatrix removing row i and column j
- Cofactor $C_{ij} = (-1)^{i+j} \cdot M_{ij}$
- $\det A = \underline{a_{i1}}C_{i1} + \underline{a_{i2}}C_{i2} + \dots + a_{in}C_{in}$ ✓ i th row

Cofactor expansion (Laplace's Theorem)



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1x1

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- Cofactor $C_{ij} = (-1)^{i+j} \cdot M_{ij}$
- $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$
- $\det A = \underline{a_{1j}}\underline{C_{1j}} + \underline{a_{2j}}\underline{C_{2j}} + \dots + \underline{a_{nj}}\underline{C_{nj}}$ j^{th} Column

Cofactor expansion (Laplace's Theorem)



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- $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$
- $\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$

The value of $\det(A)$ is the same regardless of which row or column you expand along. Choosing a row or column with many zeros minimizes computation.

Determinant of Triangular matrices



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(D) Row operations $\left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{array} \right] L$

$\det(\text{upper or lower triangular}) = \text{product of diagonal entries}$
(can also be proved with row operations)

$\det \begin{bmatrix} a_1 & a_i & a_j & a_i + a_j & a_n \end{bmatrix} \quad \begin{matrix} 1 \times 1 \\ 2 \times 2 \end{matrix}$

$$\det A = \det A^T$$

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$



$$\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} = A^T$$

$$\det(A) = \sum_j a_{1j} \cdot (-1)^{(1+j)} \cdot \det(M_{1j}(A))$$

1st row

$$\det(A^T) = \sum_j (A^T)_{j1} \cdot (-1)^{(j+1)} \cdot \det(M_{j1}(A^T))$$

1st col

$$= \sum_j a_{1j} \cdot (-1)^{(1+j)} \cdot \det(M_{j1}(A^T))$$

Some more properties



- If two rows of A are equal, then $\det A = 0$ (sign change for row swapping)

$$\begin{bmatrix} a_1 & a_1 & a_3 \\ \hline \end{bmatrix} \rightarrow \begin{bmatrix} \hline a_1 & a_1 & a_3 \end{bmatrix}$$

$$\det(A) = -\det(A)$$

Some more properties



- If two rows of A are equal, then $\det A = 0$ (sign change for row swapping)
- Subtracting a multiple of one row from another row leaves the same determinant

$$\begin{array}{c} \downarrow \\ [a_1 \quad a_i + \lambda a_j \quad a_j \quad a_n] \\ = [a_1 \quad a_i \quad a_j \quad a_n] \end{array}$$

Some more properties



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- If two rows of A are equal, then $\det A = 0$ (sign change for row swapping)
- Subtracting a multiple of one row from another row leaves the same determinant
- If A has a row of zeros, then $\det A = 0$

$$\det AB = \det A \cdot \det B$$



- Every invertible matrix is a product of elementary matrices (Gaussian elimination)

$$\det(E_1 A) = c \det(A)$$

$$= \det(E_1) \det(A) \quad \begin{pmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} E_3$$

$$\begin{matrix} -\det(A) \\ \uparrow \end{matrix}$$

$$A = LU$$

$$\det(\underline{E} A)$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{pmatrix} E_1$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} E_2$$

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- Multiplicativity holds for elementary matrices directly from the row-operation properties

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- $\det A^{-1} = \frac{1}{\det A}$

$$\det AB = \det A \cdot \det B$$



- Every invertible matrix is a product of elementary matrices (Gaussian elimination)
- Multiplicativity holds for elementary matrices directly from the row-operation properties
- $\det A^{-1} = \frac{1}{\det A}$
- $\det A$ is the product of the pivots

Next



- Eigenvectors $Ax = \lambda x$

