Matrix Theory

11 Linear Transformations

Dr. Konda Reddy Mopuri
Dept. of AI, IIT Hyderabad

July-Nov 2026

Matrices move subspaces

- Matrices are not just a bunch of numbers

Matrices move subspaces

- Matrices are not just a bunch of numbers
- They transform subspaces

Matrices move subspaces

- Matrices are not just a bunch of numbers
- They transform subspaces
- E.g., null space $\rightarrow$ zero vector and all vectors (in domain) $\rightarrow$ column space

Consider an $n \times n$ matrix A

- A transforms x to Ax

Consider an $n \times n$ matrix A

- A transforms x to Ax
- Maps the space onto itself

Example-1

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

Example-2

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

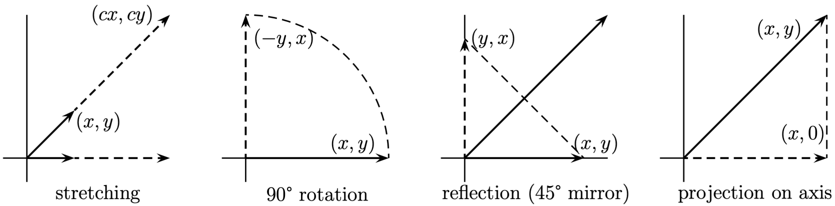
Example-3

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example-4

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Example Transformations



Example-5

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Linear Transformation

A transformation T is linear if $T(u+v) = T(u) + T(v)$
and
$T(cu) = cT(u)$.

Linear Transformation

A transformation T is linear if
$T(u+v) = T(u) + T(v)$
and
$T(cu) = cT(u)$.

- $T(c_1 u + c_2 v) = c_1 T(u) + c_2 T(v)$

Linear Transformation

A transformation T is linear if
$T(u+v) = T(u) + T(v)$
and
$T(cu) = cT(u)$.

- $T(c_1 u + c_2 v) = c_1 T(u) + c_2 T(v)$
- More generally, $T\left(\sum_i c_i u_i\right) = \sum_i c_i T(u_i)$

Linear Transformation

A transformation T is linear if
$T(u+v) = T(u) + T(v)$
and
$T(cu) = cT(u)$.

- $T(c_1 u + c_2 v) = c_1 T(u) + c_2 T(v)$
- More generally, $T\left(\sum_i c_i u_i\right) = \sum_i c_i T(u_i)$
- A linear transformation preserves linear combinations.

Example-6

- Translation: $T(x, y) = (x + 1, y)$

Example-7

- Squaring: $T(x, y) = (x^2, y)$

Matrix → Linear Transformation

- Since matrix multiplication is linear

Does every linear transformation lead to a matrix?

- Yes, once the bases are chosen

Does every linear transformation lead to a matrix?

- Yes, once the bases are chosen
- Follows nicely from the standard basis

Does every linear transformation lead to a matrix?

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- Yes, once the bases are chosen
- Follows nicely from the standard basis

> To know a linear transformation completely, it is enough to know what it does to a basis.

Example - Polynomials

- Basis for P_3: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (note that this is not unique)

Example - Polynomials

- Basis for P_3: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (note that this is not unique)
- Consider the transformation: Derivative $(\frac{\partial}{\partial t})$

Example - Polynomials

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p = 2 + t - t^2 - t^3$$

Transformation of a space to itself

- Matrix carries all the essential information.

Transformation of a space to itself

- Matrix carries all the essential information.
- If the basis is known, and the matrix is known, then the transformation of every vector is known

Transformation of a space to itself

- Matrix carries all the essential information.
- If the basis is known, and the matrix is known, then the transformation of every vector is known
- The coding of the information is simple. To transform a space into itself, one basis is enough.

Transformation of a space to itself

- Matrix carries all the essential information.
- If the basis is known, and the matrix is known, then the transformation of every vector is known
- The coding of the information is simple. To transform a space into itself, one basis is enough.
- A transformation from one space to another requires a basis for each.

Transformation between two subspaces

2U Suppose the vectors $x_1, \ldots, x_n$ are a basis for the space $\mathbf{V}$, and vectors $y_1, \ldots, y_m$ are a basis for $\mathbf{W}$. Each linear transformation T from $\mathbf{V}$ to $\mathbf{W}$ is represented by a matrix A. The jth column is found by applying T to the jth basis vector x_j, and writing $T(x_j)$ as a combination of the y's:

$$\textbf{Column } j \textbf{ of } A \qquad T(x_j) = Ax_j = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m. \qquad (6)$$

Example - Polynomials

- Basis for P_3: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (note that this is not unique)

Example - Polynomials

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- Basis for P_3: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (note that this is not unique)
- Consider the transformation: Integration $(\int_0^t)$

Example - Polynomials

- Basis for P_3: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (note that this is not unique)
- Consider the transformation: Integration $(\int_0^t)$
- Transformation from $V = P_3$ to $W = P_4$

Rotation (Q)

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Projection (P)

Reflection (H)

Summary

A matrix is not just a bunch of numbers. It represents a transformation of space. It becomes an operator acting on vectors. $T(x) = Ax$.

Basis vectors determine everything.

Next

- Orthogonality

Working Space-1

-

Working Space-2

भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad