భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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# Matrix Theory

## 11 Linear Transformations

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July-Nov 2026

# Matrices move subspaces



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- They transform subspaces
- E.g., null space $\rightarrow$ zero vector and all vectors (in domain) $\rightarrow$ column space

# Consider an $n \times n$ matrix $A$



- $A$ transforms $x$ to $Ax$

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- $A$ transforms $x$ to $Ax$
- Maps the space onto itself

# Example-1



$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

# Example-2

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

# Example-3



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

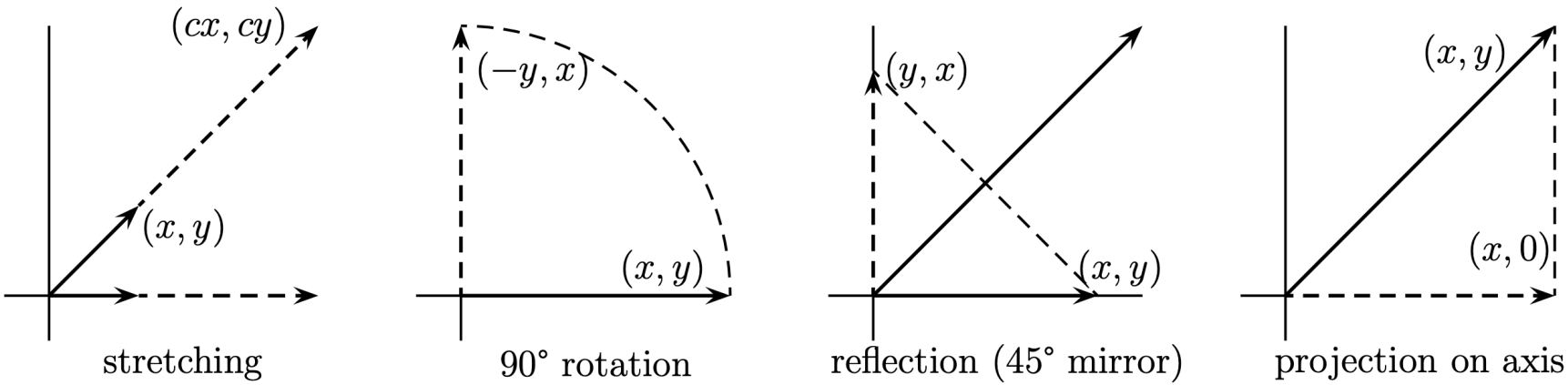
# Example-4



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

# Example Transformations





# Example-5



$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

# Linear Transformation



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- More generally, $T\left(\sum_i c_iu_i\right) = \sum_i c_iT(u_i)$
- A linear transformation preserves linear combinations.

# Example-6



- Translation: $T(x, y) = (x + 1, y)$

# Example-7



- Squaring: $T(x, y) = (x^2, y)$

# Matrix $\rightarrow$ Linear Transformation



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- $T(\alpha u + \beta v) = A(\alpha u + \beta v) = \alpha Au + \beta Av$

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To know a linear transformation completely, it is enough to know what it does to a basis.

# Example - Polynomials



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- Consider the transformation: Derivative $(\frac{\partial}{\partial t})$

# Example - Polynomials



$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$p = 2 + t - t^2 - t^3$$

# Transformation of a space to itself



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- If the basis is known, and the matrix is known, then the transformation of every vector is known
- A transformation from one space to another requires a basis for each.

# Transformation between two subspaces

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**2U** Suppose the vectors $x_1, \ldots, x_n$ are a basis for the space $\mathbf{V}$, and vectors $y_1, \ldots, y_m$ are a basis for $\mathbf{W}$. Each linear transformation $T$ from $\mathbf{V}$ to $\mathbf{W}$ is represented by a matrix $A$. The $j$th column is found by applying $T$ to the $j$th basis vector $x_j$, and writing $T(x_j)$ as a combination of the $y$'s:

$$\textbf{Column } j \textbf{ of } A \qquad T(x_j) = Ax_j = a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mj}y_m. \qquad (6)$$

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- Consider the transformation: Integration $(\int_0^t)$

# Example - Polynomials



- Basis for $P_3$: $\{p_1 = 1, p_2 = t, p_3 = t^2, p_4 = t^3\}$ (not unique!)
- Consider the transformation: Integration $(\int_0^t)$
- Transformation from $V = P_3$ to $W = P_4$

# Rotation (Q)

# Projection (P)

# Reflection (H)

# Summary



A matrix is not just a bunch of numbers. It represents a transformation of space. It becomes an operator acting on vectors. $T(x) = Ax$.

A linear transformation preserves linear combinations. $\Rightarrow T(0) = 0$.

Basis vectors determine everything.

# Next



- Orthogonality

# Working Space-1





# Working Space-2





# Working Space-3

