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Matrix Theory

09 Linear Independence, Basis, and Dimension

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From the last class

- Elimination reduces $Ax = b$ to $Ux = c$ and $Rx = d$

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Going forward

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- $m - r$ solvability constraints on $b/c/d$
- The rank of those matrices (A, U, R) is r

Rank r is crucial!

True size of the system

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

- m and n do not give the complete picture

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- Columns also failed to be independent; $C(A)$ became $2D$ plane
- Rank (true size) emerges to be an important number!

Rank: computational definition

- Number of pivots in the elimination process

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> Rank counts the number of independent rows in the matrix!

Linear Independence

Given a set of vectors, $v_1, v_2, \ldots, v_n$, and if their combination $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$ only when $c_1 = c_2 = \ldots = c_n = 0$, then the vectors $v_1, v_2, \ldots, v_n$ are linearly independent.

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And if any $c_i \neq 0$, the vectors are dependent. One vector is a combination of others!

Linear Independence

A collection is linearly independent when every vector contributes something that cannot be produced by the others.

Example-1

- If any $v_i = \bar{0}$, the set is linearly dependent

Example-2

- Columns $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Columns $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Example-3

- Columns $\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\1\\2\end{bmatrix}$

Combination $\iff$ Dependent

A set of vectors is dependent iff at least one vector can be written as a combination of the others.

Independence and Null space

- Put the vectors into the columns of a matrix $\begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \dots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$

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- $Ax = x_1 v_1 + x_2 v_2 + \ldots + x_n v_n$
- Vectors are independent $\iff$ $Ax = 0$ has only $x = 0$

Echelon Matrix

- r nonzero rows are independent

Echelon Matrix

- r nonzero rows are independent
- r columns that contain the pivots are independent

Dependent or Independent?

- A set containing 0?
- One nonzero vector?
- Two identical vectors?
- Two nonzero scalar multiples?
- More than n vectors in $\in \mathbb{R}^n$

Triangular Matrix

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$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

Identity Matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

When $m < n$

- Set of n vectors in $\mathbb{R}^m$ must be dependent

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- Set of n vectors in $\mathbb{R}^m$ must be dependent
- $A_{m \times n}$ is a fat matrix
- $Ax = 0$ has nontrivial solutions

Spanning a space

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If a vector space V consists of all linear combinations of $w_1, \ldots, w_l$, then these vectors span the space. Every vector v in V is some combination of the w's: $v = c_1 w_1 + \ldots +_l w_l$ for some c_i's.

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The c's need not be unique (different combinations are permitted; spanning set can be large!).

Example-4

- Vectors $(1,0,0)$, $(0,1,0)$, and $(-2,0,0)$ span the xy-plane in $\mathbb{R}^3$

Column space: span of columns

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- $C(A) = \{Ax : \quad x \in \mathbb{R}^n\}$

Column space: span of columns

- $C(A) = \{Ax : \quad x \in \mathbb{R}^n\}$
- Space spanned by the columns of A

Independence + Span $\rightarrow$ Basis

- Consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Independence + Span $\rightarrow$ Basis

- Consider $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- They are independent and span $\mathbb{R}^2$

Basis

A set $\{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V, if it spans V and is linearly independent.

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Basis $=$ enough vectors to generate everything, but no unnecessary vectors.

Every vector in the space is a combination of the basis vectors, because they span.

Basis expresses each vector uniquely

There is a unique way to write v as a combination of the basis vectors.

Multiple bases for the same space

- Consider $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

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- Both span $\mathbb{R}^2$. Which is correct?

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- Both span $\mathbb{R}^2$. Which is correct?
- Basis is not unique! A vector space has infinitely many different bases.

What is common across different bases of a space?

The vectors in a basis are not intrinsic to the space. But the number of vectors in every basis is intrinsic.

Dimension of a space

Any two bases for a vector space V contain the same number of vectors. This number, which is shared by all bases and expresses the number of "degrees of freedom" of the space, is the dimension of V.

Dimension is the same despite different bases!

If $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are both bases for the same vector space, then $m = n$. The number of vectors is the same.

Zero matrix is exceptional!

- Column space is $\{\bar{0}\}$

Zero matrix is exceptional!

- Column space is $\{\bar{0}\}$
- By convention, the basis is $\{\}$ and dimension is zero

Next

- The Four Fundamental Subspaces of a matrix!

Rough