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Matrix Theory

08 Complete Solution to Ax=b

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Elimination

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- We considered $Ax = b$ for square and invertible A
- Solved it $(A^{-1}b)$ via elimination
- A rectangular$A$ brings new possibilities - U may not have enough pivots
- Idea is to construct R from U, then read all the solutions from it

For an invertible matrix

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- Column space is the whole space

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- Any vector x_n in the null space can be added to a particular solution x_p
- $Ax_p = b$ and $Ax_n = 0$ produce $A(x_p + x_n) = b$

- We need to find the condition on b for which $Ax = b$ is solvable

Example-1

- 1×1 system $0x = b$

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- Solvable only if $b = 0$
- $0x = 0$ has infinitely many solutions
- Null space contains all x
- A particular solution is $x_p = 0$, and the complete solution is $x = 0 + x_n$

Example-2

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- When $b_2 = 2b_1$ (say, $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$),
- One particular solution $x_p = (1, 1)$
- Null space contains $(-1, 1)$ and all its multiples $(-c, c)$

Example-3

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

Pivots and zero rows

- Pivots are the first nonzero entries in each row
- Below each pivot is a column of zeros, obtained by elimination
- Each pivot lies to the right of the pivot in the row above $\rightarrow$ staircase pattern, and zero rows come last

PA=LU holds in $m \times n$ case too!

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From U to R

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Rx = 0$: Pivot variables and free variables

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- General solution to $Rx = 0 \leftarrow$ arbitrary values to the free variables

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Finding all solutiosn to $Ax = 0$

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Null space from Special solutions

- Reach $Rx = 0$, identify the pivot and free variables
- Special solutions: give one free variable the value 1 (other free variables to 0), and solve $Rx = 0$, for the pivot variables.
- Find all special solutions. Combinations of special solutions is all solutions to Null space.

Special solutions are visible in R

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- $Ax = 0$ has at least one special solution
- There are more solutions than the trivial $x = 0$.

Solving $Ax = b$

- $b \neq 0$, Row operations have to be applied on RHS

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_1 + 5b_1 \end{bmatrix}$$

Solving $Ax = b$

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- $\rightarrow Ux = c$

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- Let's choose $b = (1, 5, 5)$

Solving Ax=b: back-substitution

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

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- Comes from solving the equation with all free variables set to zero

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- Is it a subspace?

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- Find all the special solutions to $Ax = 0$

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- Then $x = x_p +$ (any combination x_n of special solutions)

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- Reduce $Ax = b$ to $Ux = c$
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- Then $x = x_p +$ (any combination x_n of special solutions)
- R make this solution even clearer

Next

- Linear Independence, Basis, and Dimension