Matrix Theory

08 Complete Solution to Ax=b

Dr. Konda Reddy Mopuri
Dept. of AI, IIT Hyderabad
July-Nov 2026

Elimination

- We considered $Ax = b$ for square and invertible A

Elimination

- We considered $Ax = b$ for square and invertible A
- Solved it $(A^{-1}b)$ via elimination

Elimination

'n' pivots $\iff$ invert-ible

- We considered $Ax = b$ for square and invertible A
- Solved it $(A^{-1}b)$ via elimination
- A rectangular A brings new possibilities - U may not have enough pivots

Elimination

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- We considered $Ax = b$ for square and invertible A
- Solved it $(A^{-1}b)$ via elimination
- A rectangular$A$ brings new possibilities - U may not have enough pivots
- Idea is to construct R from U, then read all the solutions from it

For an invertible matrix

- Null space is $\{\bar{0}\}$

For an invertible matrix

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- Null space is $\{\bar{0}\}$
- Column space is the whole space

$Ax = 0$

$A^{-1}Ax = A^{-1} \cdot 0$

$x = 0$

$Ax = \boxed{b \in \mathbb{R}^n}$

$x = A^{-1}b$

When the null space is more than zero vector

- Any vector x_n in the null space can be added to a particular solution x_p

When the null space is more than zero vector

- Any vector x_n in the null space can be added to a particular solution x_p
- $Ax_p = b$ and $Ax_n = 0$ produce $A(x_p + x_n) = b$

and this is a solution too!

When the column space is less than all vectors

$b \in \mathbb{R}^m$

$A_{m \times n}$

$C(A) \subseteq \mathbb{R}^m$

- We need to find the condition on b for which $Ax = b$ is solvable

$\boxed{A \quad \underline{b}}$ solvable

$\underline{b}$

not solvable

Example-1

- 1×1 system $0x = b$

Example-1

- 1×1 system $0x = b$
- Solvable only if $b = 0$

Example-1

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- 1×1 system $0x = b$
- Solvable only if $b = 0$
- $0x = 0$ has infinitely many solutions $x \in \mathbb{R}$

Example-1

- 1×1 system $0x = b$
- Solvable only if $b = 0$
- $0x = 0$ has infinitely many solutions
- Null space contains all x

$N(A)$

one particular solution $x_p = 0$

$x_n \in \mathbb{R}$

$0(x_p + x_n) = 0$

Example-1

- 1×1 system $0x = b$
- Solvable only if $b = 0$
- $0x = 0$ has infinitely many solutions
- Null space contains all x
- A particular solution is $x_p = 0$, and the complete solution is $x = 0 + x_n$

Example-2

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}_{2\times 2} \xrightarrow{E} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

- is not invertible

Example-2

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

- is not invertible
- There is no solution unless $b_2 = 2b_1$

$$Ax = b$$

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix}$$

$$Col(A) \subseteq \mathbb{R}^2$$

$$Ax = b \xrightarrow[E]{} Ux = c$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \end{bmatrix}$$

$$b_2 - 2b_1 = 0 \Rightarrow \boxed{b_2 = 2b_1}$$

Consistent

Example-2

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- is not invertible
- There is no solution unless $b_2 = 2b_1$
- When $b_2 = 2b_1$ (say, $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$),

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Example-2

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

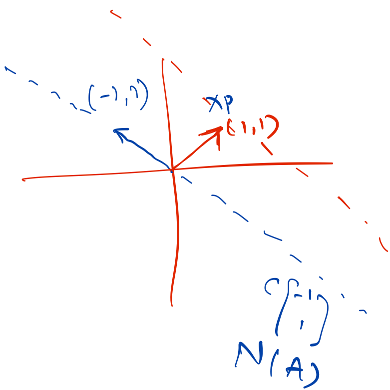
$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- is not invertible
- There is no solution unless $b_2 = 2b_1$
- When $b_2 = 2b_1$ (say, $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$),
- One particular solution $x_p = (1, 1)$

$x_n \in N(A)$

$\underbrace{(-1, 1)}_{x_n}$ $(-c, c)$

Example-2



$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

- is not invertible
- There is no solution unless $b_2 = 2b_1$
- When $b_2 = 2b_1$ (say, $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$),
- One particular solution $x_p = (1, 1)$
- Null space contains $(-1, 1)$ and all its multiples $(-c, c)$

b

$c\begin{bmatrix} -1 \\ 1 \end{bmatrix} \subseteq \mathbb{R}^2$ $\qquad x_p + \{x_n\}$ $\qquad Ax = 0$

Example-3

a_{ij}

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

3×4

$R_2: -2R_1 + R_2$

$R_3: R_1 + R_3$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix}$$

$m \times n$

$R_3: -2R_2 + R_3$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

U

$Ax = 0$

$Ux = 0$

Pivots and zero rows

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

- Pivots are the first nonzero entries in each row
- Below each pivot is a column of zeros, obtained by elimination
- Each pivot lies to the right of the pivot in the row above $\rightarrow$ staircase pattern, and zero rows come last

PA=LU holds in $m \times n$ case too!

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix}$$

$$\boxed{G\ F\ E}\ \underset{m\times n}{A} = \boxed{U_{m \times n}}$$

$$L^{-1}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}$$

From U to R

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_2 : R_2/3]{} \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1 : R_1 - 3R_2]{}$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$Ax = 0$

$\boxed{x_p + x_n}$

pivot columns $\longrightarrow$ pivot variables $\begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix}$

[rest are free columns] [rest are free variables]

$Rx = 0$: Pivot variables and free variables

$$A = \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Special solution 1
special solution 2

- General solution to $Rx = 0 \leftarrow$ arbitrary values to the free variables

Back substitution

$w + y = 0 \Rightarrow w = -y$

$u + 3v - y = 0 \Rightarrow u = -3v + y$

$$\Rightarrow x = \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -3v+y \\ v \\ -y \\ y \end{bmatrix} = \underline{v} \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \underline{y} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Solution to $Ax = 0$ is a 2D subspace in $\mathbb{R}^4$

These special solutions are visible in R. Look at the free columns.

$1.u + 3.v + 0.w - 1.y = 0$

$0.u + 0.v + 1.w + 1.y = 0$

$\Leftarrow$

$u + 3 = 0$, $w = 0$ | $u - 1 = 0$, $w + 1 = 0$

(v, w) 1 0 | 0 1

Finding all solutiosn to $Ax = 0$

Null space from Special solutions

- Reach $Rx = 0$, identify the pivot and free variables

Finding all solutiosn to $Ax = 0$

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Null space from Special solutions

- Reach $Rx = 0$, identify the pivot and free variables
- Special solutions: give one free variable the value 1 (other free variables to 0), and solve $Rx = 0$, for the pivot variables.

Finding all solutiosn to $Ax = 0$

Null space from Special solutions

- Reach $Rx = 0$, identify the pivot and free variables
- Special solutions: give one free variable the value 1 (other free variables to 0), and solve $Rx = 0$, for the pivot variables.
- Find all special solutions. Combinations of special solutions is all solutions to Null space.

$\rightarrow$ 1 0 0 $\rightarrow$ SP1
$\rightarrow$ 0 1 0 $\rightarrow$ SP2
0 0 1 $\rightarrow$ SP3

Special solutions are visible in R

More variables than equations

Indian Institute of Technology Hyderabad

$Ax = b$

$A_{m \times n}$

- If $m < n$, there must be at least $n - m$ free variables

More variable than equations

m

More variables than equations

- If $m < n$, there must be at least $n - m$ free variables
- $Ax = 0$ has at least one special solution

More variables than equations

- If $m < n$, there must be at least $n - m$ free variables
- $Ax = 0$ has at least one special solution
- There are more solutions than the trivial $x = 0$.

Solving $Ax = b$

- $b \neq 0$, Row operations have to be applied on RHS

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{bmatrix}$$

$U \quad 3 \times 4 \qquad\qquad C$

$x_p + x_n$

$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$b_3 - 2b_2 + 5b_1 = 0$

$C(A)$

Solving $Ax = b$

- $b \neq 0$, Row operations have to be applied on RHS
- $\rightarrow Ux = c$

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_1 + 5b_1 \end{bmatrix}$$

- Let's choose $b = (1, 5, 5)$

Solving Ax=b: back-substitution

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

$$\xrightarrow{R2:\ R2/3} \left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R1:\ R1-3R_2}$$

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$$R \quad x = d$$

$$\begin{array}{ccc} & sp1 & & sp2 \\ x_n = v & \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} & + \; y & \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \end{array}$$

$$\underline{x_p} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Particular Solution

- Comes from solving the equation with all free variables set to zero

$$x_{com} = \underline{x_{par}} + x_{sp}$$

$N(A)$

Particular Solution

- Comes from solving the equation with all free variables set to zero
- Is it a subspace?

Solutions to Ax=b

$x_{complete}$

- Reduce $Ax = b$ to $Ux = c$

Solutions to Ax=b

$x_{complete}$

- Reduce $Ax = b$ to $Ux = c$
- With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$

Solutions to Ax=b

$x_{complete}$

- Reduce $Ax = b$ to $Ux = c$
- With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$
- Find all the special solutions to $Ax = 0$

Solutions to Ax=b

$x_{complete}$

- Reduce $Ax = b$ to $Ux = c$
- With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$
- Find all the special solutions to $Ax = 0$
- Then $x = x_p +$ (any combination x_n of special solutions)

Solutions to Ax=b

भारतीय सांकेतिक विज्ञान संस्थान हैदराबाद
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

$Rx = d$ has everything

$x_{complete}$

- Reduce $Ax = b$ to $Ux = c$
- With free variables $= 0$, find a particular solution to $Ax_p = b$ and $Ux_p = c$
- Find all the special solutions to $Ax = 0$
- Then $x = x_p +$ (any combination x_n of special solutions)
- R make this solution even clearer

"read out"

free $\longrightarrow$ dependent

Next

- Linear Independence, Basis, and Dimension