Matrix Theory

07 Column and Null Spaces of a Matrix

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Column Space

- All linear combinations of columns of a matrix

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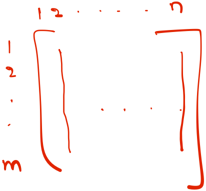
- Consider the column picture of $A \in \mathbb{R}^{m \times n}$



Column Space

- All linear combinations of columns of a matrix
- Consider the column picture of $A \in \mathbb{R}^{m \times n}$
- Which all $b's$ can A produce?

$$\underbrace{A}\, x = b$$

$$\underset{m}{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}} \cdot \underset{v}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} =$$

Column Space

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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- All linear combinations of columns of a matrix
- Consider the column picture of $A \in \mathbb{R}^{m \times n}$
- Which all $b's$ can A produce?
- $Col(A) = \{Ax : x \in \mathbb{R}^n\}$

Column Space

$A_{m \times n}$

- $Col(A) \subseteq \mathbf{R}^m$

Column Space

$$C\left(\begin{bmatrix}1 & 7\\ 2 & 8\\ 3 & 9\end{bmatrix}\right) \subseteq \mathbb{R}^3$$

$A_{3\times 2}$

- $Col(A) \subseteq \mathbf{R}^m$
- $Ax = b$ has a solution $\iff b \in Col(A)$

$$C\left(\begin{bmatrix}a_1 & a_2 & | & | & |\end{bmatrix}_{3\times 5}\right) \subseteq \mathbb{R}^3$$

Is Column space a subspace?

$A_{m \times n}$

$\text{Col}(A)$

✓ If b, then $-b$ $[\because A(-x)]$

✓ $\bar{0} \in \text{Col}(A)$

✓ If $\{u, v\} \Longrightarrow (\alpha u + \beta v) \subseteq C(A)$

x_1 x_2 αx_1 βx_2

What does A send to zero?

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$$A_{m\times n}\, x_{n\times 1} = \begin{matrix}0\\0\\0\end{matrix}_{m\times 1}$$

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What does A send to zero?

- For an $m \times n$ matrix A, which i/p vectors disappear when multiplied by A? i.e., $Ax = 0$
- $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$
- Null space or Kernel of A
- $N(A) \subseteq \mathbb{R}^n$

Is Null space a subspace?

$x = 0$

$Ax \longrightarrow 0$

$u, \quad v$

$Au = 0 \quad Av = 0$

$\Longrightarrow$

$(\alpha u + \beta v)$

$A(\alpha u + \beta v)$

$\alpha Au + \beta Av$

$= \alpha \bar{0} + \beta \bar{0}$

$= \bar{0}$

Input-Output Interpretation

Vectors in Vector Space

$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$

- $Col(A)$ contains all possible outputs.

$$A_{m\times n} \, \big|_{n\times 1} = \big|_{m\times 1}$$

$$\mathbb{R}^n \qquad \mathbb{R}^m$$

Input-Output Interpretation

Vectors in Vector Space

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^m$$

- $Col(A)$ contains all possible outputs.
- $N(A)$ contains all inputs producing zero output.

Effect of Row Operations

- $A \rightarrow U \rightarrow R$

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- $A \to U \to R$
- $N(A) = N(U) = N(R)$

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- $A \rightarrow U \rightarrow R$
- $N(A) = N(U) = N(R)$
- $C(A) \neq C(U) \neq C(R)$

Summary

Column space and Null space

Column space describes what A can produce; null space describes what A cannot distinguish from zero.

Next

- How to find every solution of $Ax = b$?