Matrix Theory

06 Vector Spaces and Subspaces

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- Helps us with the basic questions of *existence* and *uniqueness*
- How to find every solution?

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- Addition and scalar multiplication

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- Add them: $p(x) + q(x)$
- Multiply by scalars: $3p(x)$

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- Matrices can be added and multiplied by scalars

Do vectors need to be columns of numbers?

In linear algebra, a vector is not necessarily an arrow or a column of numbers. A vector is an object belonging to a set in which addition and scalar multiplication behave in the familiar linear way.

Vector Space

Let V be a nonempty set, and let the scalars come from a field $\mathbb{F}$, (usually $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

We have two operations:
$+ : V \times V \to V$, called vector addition, and
$\cdot : \mathbb{F} \times V \to V$, called scalar multiplication.
We call V a vector space over $\mathbb{F}$ if, for every $u, v, w \in V$ and every $a, b \in \mathbb{F}$, the following properties hold:

Vector Space

1. Commutativity of addition
2. Associativity of addition
3. Existence of a zero vector
4. Existence of additive inverses
5. Distributivity over vector addition
6. Distributivity over scalar addition
7. Compatibility of scalar multiplication
8. Identity scalar

Vector Space

There is an important point: $u + v \in V$ and $av \in V$.
That is, the set is closed under addition and scalar multiplication.

Vector Space

Addition behaves normally

$u + v = v + u$, $(u + v) + w = u + (v + w)$, $v + 0 = v$, $v + (-v) = 0$

Scalar Multiplication behaves normally

$1v = v$, $a(bv) = (ab)v$

Addition and Scalar Multiplication interact normally

$a(u + v) = au + av$, $(a + b)v = av + bv$

Vector Space

Informally

A vector space is a collection of objects in which linear combinations make sense and stay inside the collection.
If, $v_1, v_2, \ldots v_k \in V$, and
$c_1, c_2, \ldots, c_k \in \mathbb{F}$, then $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k \in V$.

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Vectors in Vector Space

These objects look completely different - columns, matrices, polynomials, functions - but from the viewpoint of linear algebra, they are all the same kind of object.

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- Largest subspace of a space?

Subspace

- Consider $\mathbb{R}^3$
- Some subspaces in this space:

Subspace

- Consider $\mathbb{R}^2$

Subspace

- Consider $\mathbb{R}^2$
- Is the first quadrant a subspace?

Next

- Column space and Null space of a matrix