Matrix Theory

03 Triangular Factorization

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Matrix notation

- For smaller systems it is easy to list the elimination steps

Matrix notation

- For smaller systems it is easy to list the elimination steps
- For larger systems, matrix multiplication can describe them concisely

Matrix Notation

- $Ax = b$

Matrix Notation

- $Ax = b$
- A - Coefficient Matrix

Matrix Notation

- $Ax = b$
- A - Coefficient Matrix
- x - unknowns
- b - RHS vector

Matrix Notation

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$$
\begin{aligned}
2u + v + w &= 5 \\
4u - 6v &= -2 \\
-2u + 7v + 2w &= 9
\end{aligned}
$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$Ax = b$

$$x = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$b = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Matrix Addition

- Both the matrices have the same shape

Matrix Addition

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- Both the matrices have the same shape
- Added one entry at a time

Matrix Addition

- Both the matrices have the same shape
- Added one entry at a time
- Results in the same shape

Multiplication by a scalar

- Done one element at a time
- Results in the same shape

Matrix Vector Multiplication

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

- System is expressed as a matrix times vector

$2u + v + w = 5$

Matrix Vector Multiplication

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

- System is expressed as a matrix times vector
- Row times a column: called 'inner product'

Matrix Vector Multiplication

a_{ij}

i^{th} row j^{th} col

- a_{ij} - elements of matrix A
- x_j - elements of column x
- i^{th} component of the product $b_i =$ $\sum_j a_{ij} x_j$

$\boxed{a_{i:}}\ x$

$\langle a_{i:}, x \rangle$

MV product - Combining columns

- Combine the columns of matrix as per the entries in the vector

$$\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} = 2\underbrace{\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}}_{a_{1:}} + 5\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{a_{2:}} + 0\underbrace{\begin{bmatrix} 6 \\ 1 \\ 0 \end{bmatrix}}_{a_{3:}}$$

$$= \begin{bmatrix} 7 \\ 6 \\ 7 \end{bmatrix}$$

Elimination step - Matrix notation

$$\begin{array}{ccccc} A & & x & & b \\ \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} & \cdot & \begin{bmatrix} u \\ v \\ w \end{bmatrix} & = & \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \end{array}$$

- Step-1 was to eliminate 4

$R_2 : R_2 - 2R_1$

Elimination step - Matrix notation

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$$\left(X \right)\overbrace{\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}}^{A} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = (X)\begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

- Step-1 was to eliminate 4
- Since we only know MV product, let's work on the RHS

R2: R2 - 2R1

$E_{31} = 1$

$R_3 : R_3 + 1 \cdot R1$

b

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & \cancel{7} & 2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ -2-10 \\ 9 \end{bmatrix}$$

$E_{32} : 1$

3

- Step-1 was to eliminate 4
- Since we only know MV product, let's work on the RHS
- It can be achieved by multiplying the RHS with an elementary/elimination matrix E

$$E_{21} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 9 \end{bmatrix}$$

R2 : R2 - 2R1 + OR3

$E_{ij} = -l$

$R_i = \boxed{R_i - l R_j}$

Elimination step - Matrix notation

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- An identity matrix leaves RHS (b) unchanged

Elimination step - Matrix notation

- An identity matrix leaves RHS (b) unchanged
- Elimination matrix $E_{ij} = -l$ subtracts l times R_j from R_i

Elimination step - Matrix notation

- What happens to LHS?

Elimination step - Matrix notation

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॥ नवोन्मेषेण देश निर्माणम् ॥

M.V.

- What happens to LHS?
- We have to multiply by the same $E \rightarrow$ Matrix Multiplication

Matrix Multiplication

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- To multiply two matrices A, B:
  - if square: should have the same size
  - if rectangular: no. of columns in the left matrix should be same as the no. of rows in the right one

Matrix Multiplication

- To multiply two matrices A, B:
  - if square: should have the same size
  - if rectangular: no. of columns in the left matrix should be same as the no. of rows in the right one
- If A is $m \times \cancel{n}$ and B is $\cancel{n} \times p$, then AB will be of size $m \times p$

Matrix Multiplication

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$_{m \times n}A = [a_{ij}]$
$_{n \times p}B = [b_{ij}]$

1. Inner product: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$

Matrix Multiplication

$$A \cdot \begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$$

1. Inner product: $\boxed{(AB)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}}$
2. MV product: Each column of (AB) is the product of A and individual columns of B

$$\begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

Matrix Multiplication

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1. Inner product: $(AB)_{ij} = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$
2. MV product: Each column of (AB) is the product of A and individual columns of B
3. Row-Matrix product: Each row of (AB) is the product of a row from A and B

Matrix Multiplication

- Not Commutative $AB \neq BA$

Matrix Multiplication

- Not Commutative
- Associative $(AB)C = A(BC)$

Matrix Multiplication

- Not Commutative
- Associative
- Distributive

$$A(B+C) = AB + AC$$
$$(A+B)C = AC + BC$$

Triangular Factors: A to U

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} A$$

$R3: R3 + R2$ $\quad$ $R3: R1 + R3$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \begin{matrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & \cancel{8} & 3 \end{matrix} \leftarrow \begin{matrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ -2 & 7 & 2 \end{matrix}$$

U

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

GFE

$\boxed{GFE}\,A = U$

$\boxed{GFE}\,b = c$

Triangular Factors: U to A

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow F^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow E^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R3 : R3 − R2   R3 : R3 − R1

$$E^{-1} \times F^{-1} \times G^{-1}\, U = A$$

$$E^{-1}F^{-1}G^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \xrightarrow{E^{-1}} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \boxed{E^{-1}F^{-1}G^{-1}}\, U$$

$$LU = A$$

Example-1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$$

$n^3/3$

$(GFE)Ax =$

$(GFE)b$

$Ax =$

$Ux = c$

Gauss Elim

$Ax = b$

$Ux = c$

$Lc = b$

$Lc = b$

$[L]Ux = Ax$

$Lc = b$

Example-2

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{matrix} \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{matrix} \quad \begin{matrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\boxed{\begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{matrix}} = \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{matrix}}_{G} \quad \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}}_{F} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{E}$$

Rough

-

Rough