భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad
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Matrix Theory

02 Gaussian Elimination

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Example system of equations

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- Let's solve this!

An intuitive approach

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad\;\; &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- Start subtracting multiples of the first equation from the rest

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- Start subtracting multiples of the first equation from the rest
- Goal: simplify the system via eliminating (or reducing) the variables

An intuitive approach

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- $R2 \leftarrow R2 - 2 \cdot R1$

An intuitive approach

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

- $R2 \leftarrow R2 - 2 \cdot R1$
- $R3 \leftarrow R3 + 1 \cdot R1$

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned}$$

The Intuitive Approach

$$2u + v + w = 5$$
$$-8v - 2w = -12$$
$$8v + 3w = 14$$

- We proceed to eliminate another variable (v in this case)

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- This time, we need not work with the first equation
- Instead, we start from the second

The Intuitive Approach

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- We proceed to eliminate another variable (v in this case)
- This time, we need not work with the first equation
- Instead, we start from the second
- $R3 \leftarrow R3 + R2$

The Intuitive Approach

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ 8v + 3w &= 14 \end{aligned}$$

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- This process is called 'Gaussian Elimination' or 'Elimination'

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- This process is called 'Gaussian Elimination' or 'Elimination'
- Specifically, the 'Forward' part of the elimination

Elimination

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

- Goal of Forward part: (Upper) Triangular system

Elimination

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

- Goal of Forward part: (Upper) Triangular system
- Known as 'Row Echelon Form (REF)'

Elimination

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

- Goal of Forward part: (Upper) Triangular system
- Known as 'Row Echelon Form (REF)'
- Coefficient 2 in $R1$ and -8 in $R2$ are known as 'pivots'

Elimination

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

- Goal of Forward part: (Upper) Triangular system
- Known as 'Row Echelon Form (REF)'
- Coefficient 2 in $R1$ and -8 in $R2$ are known as 'pivots'
- Pivots find the right multipliers to eliminate (hence can't be zeros)

Elimination

$$
\begin{aligned}
2u + v + w &= 5 \\
-8v - 2w &= -12 \\
w &= 2
\end{aligned}
$$

- Triangular system can be solved backward

Elimination

$$\begin{aligned} 2u + v + w &= 5 \\ -8v - 2w &= -12 \\ w &= 2 \end{aligned}$$

- Triangular system can be solved backward
- This part is called 'back substitution'

Elimination: Summary

Forward elimination produces pivots. It subtracts multiples of each row from the rows below it. This gives us a triangular system. We solve it in the reverse order, substituting each newly computed variable into the equations that are above.

Elimination

- One way to write down the forward elimination is by including the RHS vector

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

Elimination

- One way to write down the forward elimination is by including the RHS vector
- Perform the same operations on the RHS

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{array}\right]$$

$R_2 : R_2 - 2R_1$

$R_3 : R_3 + R_1$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$R_3 : R_3 + R_2$

The breakdown of Elimination

- When will this process fail?

The breakdown of Elimination

- When will this process fail?
- When you can't perform the forward or backward operations

The breakdown of Elimination

- Full set of n pivots $\rightarrow$

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- Only one solution; systems is nonsingular

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- If zero appears in pivot position $\rightarrow$

The breakdown of Elimination

- Full set of n pivots $\rightarrow$
- Only one solution; systems is nonsingular
- If zero appears in pivot position $\rightarrow$
- System might or might not be singular

The breakdown of Elimination

- Many times a swap of rows can fix this $\rightarrow$ nonsingular

The breakdown of Elimination

- Many times a swap of rows can fix this $\rightarrow$ nonsingular
- Some times breakdown is unavoidable $\rightarrow$ singular

Example - 1

$$u + v + w = \_\_$$
$$2u + 2v + 5w = \_\_$$
$$4u + 6v + 8w = \_\_$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

$R2 : R2 - 2R1$
$R3 : R3 - 4R1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix}$$

swap (R2, R3)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

Example - 2

$$u + v + w = \_\_$$
$$2u + 2v + 5w = \_\_$$
$$4u + 4v + 8w = \_\_$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix} \xrightarrow[R3: R3-4R1]{R2: R2-2R1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$R3: R3 - \frac{4}{3} R2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & - \\ 0 & 0 & 3 & - \\ 0 & 0 & 0 & ? \end{array}\right]$$

Zero → Consistent [many solutions]

Nonzero → Inconsistent [No solution]

Cost of Elimination

- How many separate arithmetic operations are required for solving a system with n equations in n variables?

Cost of Elimination

- Let's ignore the RHS for now

Cost of Elimination

- Let's ignore the RHS for now
- One 'division' is one operation ✓

Cost of Elimination

- Let's ignore the RHS for now
- One 'division' is one operation
- One 'Multiply-subtract' one operation

Cost of Elimination: first column

- Step 1: find the right multiplier for each equation via division by the pivot

$(n-1)$ rows below R_1

$\Rightarrow$ $(n-1)$ multipliers are needed

$\Rightarrow$ $(n-1)$ division operations

Cost of Elimination: first column

- Step 1: find the right multiplier for each equation via division by the pivot
- Step 2: perform 'multiply and subtract' operation

Each row needs (n-1) coefficient updates

There are (n-1) rows below R1

$\Rightarrow$ (n-1).(n-1) M&S operations

Cost of Elimination: first column

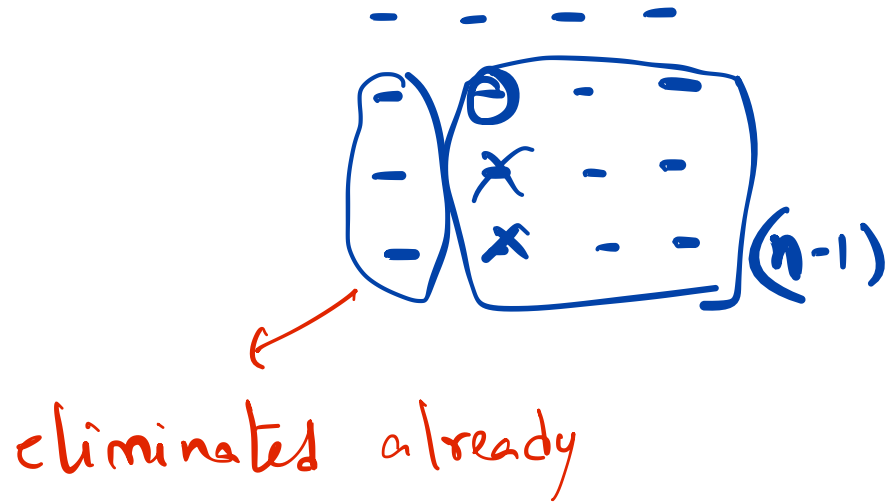
- Step 1: find the right multiplier for each equation via division by the pivot
- Step 2: perform 'multiply and subtract' operation
- $(n-1)$ 'division' operations and $(n-1)\cdot(n-1)$ 'multiply-subtract' operations

Cost of Elimination: first column

- Step 1: find the right multiplier for each equation via division by the pivot
- Step 2: perform 'multiply and subtract' operation
- $(n-1)$ 'division' operations and $(n-1)\cdot(n-1)$ 'multiply-subtract' operations
- Total $n(n-1)$ operations

Cost of Elimination: second column

- Equations become shorter



Cost of Elimination: second column

- Equations become shorter
- Step 1: find the right multiplier for each equation via division by the second pivot

Cost of Elimination: second column

- Equations become shorter
- Step 1: find the right multiplier for each equation via division by the second pivot
- Step 2: perform 'multiply and subtract' operation

Cost of Elimination: second column

- Equations become shorter
- Step 1: find the right multiplier for each equation via division by the second pivot
- Step 2: perform 'multiply and subtract' operation
- $(n-2)$ 'division' operations and $(n-2)\cdot(n-2)$ 'multiply-subtract' operations

Cost of Elimination: second column

- Equations become shorter
- Step 1: find the right multiplier for each equation via division by the second pivot
- Step 2: perform 'multiply and subtract' operation
- $(n-2)$ 'division' operations and $(n-2)\cdot(n-2)$ 'multiply-subtract' operations
- Total $(n-1)(n-2)$ operations

Cost of Elimination: LHS

- $n \cdot (n-1) + (n-1) \cdot (n-2) + \ldots + 2.(2-1)$

$= n^2 - n + (n-1)^2 \cdot (n-1) + \ldots + 2^2 - 2 + 1^2 - 1$

$= (n^2 + (n-1)^2 + \ldots 1^2) - (n + (n-1) \ldots + 1)$

$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$

$O(n^3)$

$= n\left(\frac{n+1}{2}\right)\left[\frac{1}{3}(2n+1) - 1\right] = \frac{2(n-1)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n^2-1)}{3}$

Cost of Elimination: RHS

- First column:

Cost of Elimination: RHS

- First column: $(n-1)$ 'multiply-subtract' operations

Cost of Elimination: RHS

- First column: $(n-1)$ 'multiply-subtract' operations
- Second column:

Cost of Elimination: RHS

- First column: $(n-1)$ 'multiply-subtract' operations
- Second column: $(n-2)$ 'multiply-subtract' operations

Cost of Elimination: RHS

- First column: $(n-1)$ 'multiply-subtract' operations
- Second column: $(n-2)$ 'multiply-subtract' operations
- Total $=$

Cost of back substitution

- Last equation:

Cost of back substitution

- Last equation: one division

Cost of back substitution

- Last equation: one division
- Second from the last:

Cost of back substitution

- Last equation: one division
- Second from the last: one multiply-subtract and one division

Cost of back substitution

- Last equation: one division
- Second from the last: one multiply-subtract and one division
- Total:

Can we simplify the Gaussian elimination further?

$$x + 2y = 5$$
$$2x + 3y = 8$$

Rough work