భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Matrix Theory

01 System of Linear Equations

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July-Nov 2026

The central question

What does a system of linear equations mean geometrically, algebraically, and computationally?

Let's work with an example

$$
\begin{aligned}
x + 2y &= 5 \\
3x - y &= 1
\end{aligned}
$$

- What does it mean to solve the system?

Let's work with an example

$$x + 2y = 5$$
$$3x - y = 1$$

- What does it mean to solve the system?
- Find the values of x and y that satisfy both the equations

The row picture

$$x + 2y = 5 \qquad 1$$
$$3x - y = 1 \qquad 2$$

- Consider each equation separately

The row picture

$$x + 2y = 5$$
$$3x - y = 1$$

- Consider each equation separately
- What do they represent?

The row picture

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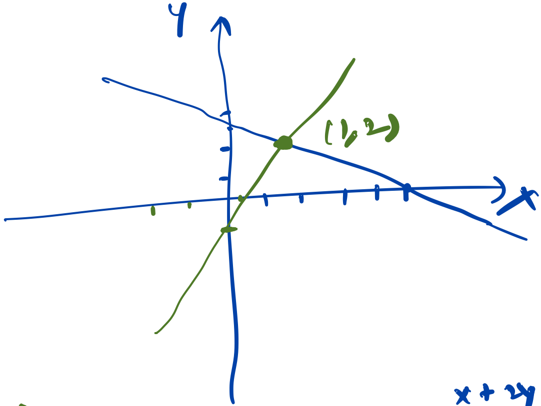
$$x + 2y = 5$$
$$3x - y = 1$$

- Consider each equation separately
- What do they represent? line in xy-plane
- What is the solution? point of Intersection

The row picture - let's solve geometrically



$3x - y = 1$

$(0,-1) \quad (1/3, 0)$

$x + 2y = 5$

$(0, 5/2)$

$(5, 0)$

The row picture

- What are the possibilities for 2 variable case?

The row picture

- What are the possibilities for 2 variable case?
  - Lines intersect: Unique solution | Consistent

The row picture

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- What are the possibilities for 2 variable case?
  - Lines intersect:
  - Distinct parallel lines: no solution / Inconsistent
    singular

The row picture

- What are the possibilities for 2 variable case?
  - Lines intersect:
  - Distinct parallel lines:
  - Lines coincide: Infinite solution / Singular

The row picture

$$\begin{aligned} x + 2y + z &= 4 \\ 2x - y + z &= 1 \\ x + y - z &= 2 \end{aligned}$$

- Let's consider 3 variables!

The row picture

$$
\begin{aligned}
x + 2y + z &= 4 \\
2x - y + z &= 1 \\
x + y - z &= 2
\end{aligned}
$$

- Let's consider 3 variables!
- What does each equation represent?

The row picture

$$
\begin{aligned}
x + 2y + z &= 4 \\
2x - y + z &= 1 \\
x + y - z &= 2
\end{aligned}
$$

- Let's consider 3 variables!
- What does each equation represent?
- How do they intersect?

The row picture

- What are the possibilities for the 3 variable case?

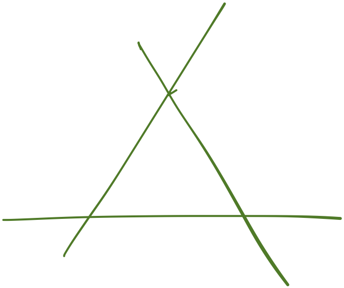
$$
\begin{aligned}
u + v + w &= 2 \\
2u + \quad\;\; 3w &= 5 \\
3u + v + 4w &= 6
\end{aligned}
$$

The row picture

- What are the possibilities for the 3 variable case?
- Case 1: Planes not parallel but don't intersect (inconsistent!)

$$\begin{aligned} u + v + w &= 2 \quad \times 1 \\ 2u + \quad 3w &= 5 \quad \times 1 \\ 3u + v + 4w &= 6 \quad \times -1 \end{aligned}$$

$$0 = 1$$



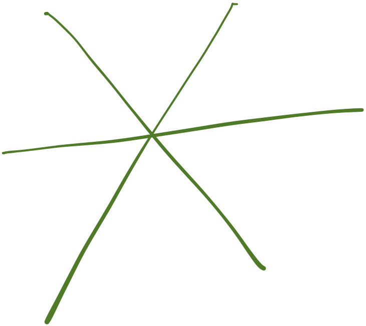
The row picture

- Case 2: Planes intersect in a line (infinite solutions!)

$$\begin{aligned} u + v + w &= 2 \\ 2u + \quad 3w &= 5 \\ 3u + v + 4w &= 7 \end{aligned}$$

$0 = 0$

consistent but singular



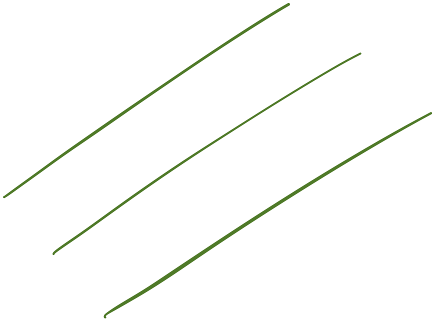
The row picture

- Case 3: Planes are parallel and distinct (inconsistent or no solutions!)

$$u + v + w = 2 \qquad \times 0$$
$$u + v + w = 5 \qquad \times 1$$
$$u + v + w = 7 \qquad \times -1$$

$$0 = -2$$

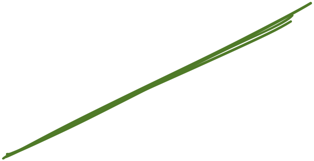
inconsistent



The row picture

- Case 4: Planes coincide (infinite solutions!)

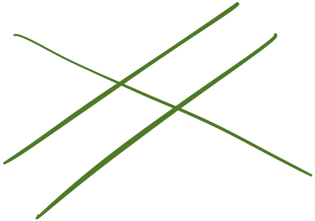
$$
\begin{aligned}
u + v + w &= 2 \\
2u + 2v + 2w &= 4 \\
3u + 3v + 3w &= 6
\end{aligned}
$$



The row picture

- Case 5: Two planes are parallel and the third is not (no solution!)

$$u + v + w = 2$$
$$2u + 2v + 2w = 5$$
$$3u + v + 4w = 7$$



The row picture

The row picture is geometrically intuitive in two and three dimensions, but becomes difficult to visualize in higher dimensions.

The column picture

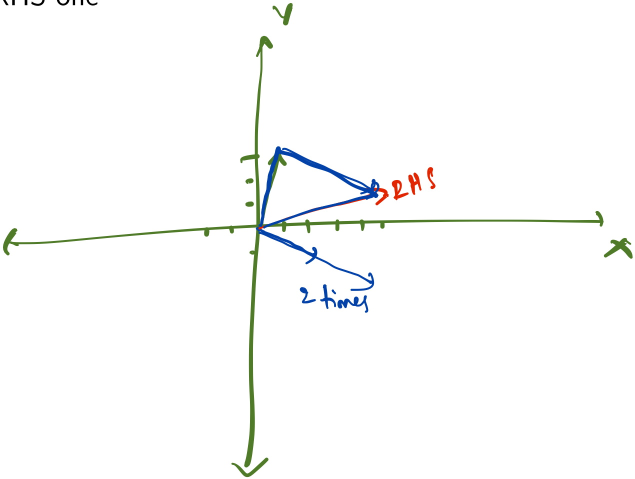
$$x + 2y = 5$$
$$3x - y = 1$$

- The two separate equations are one vector equation!

$$x \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The column picture

- The problem now becomes to find the combination of LHS vectors that gives RHS one



The column picture

- Each vector is a point (Descartes turned geometry into algebra!)

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- Basic operations on vectors: multiplying by numbers and adding vectors

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- Each vector is a point (Descartes turned geometry into algebra!)
- Basic operations on vectors: multiplying by numbers and adding vectors
- The result of multiplying vectors by scalars and adding is called 'Linear combination'

The column picture

The column picture attempts to find the right linear combination of LHS vectors to produce the one on the RHS.

The column picture

- When will this fail?

$$\begin{aligned} u + v + w &= b_1 \\ 2u + \quad\ 3w &= b_2 \\ 3u + v + 4w &= b_3 \end{aligned}$$

The column picture

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$$\begin{aligned} u + v + w &= b_1 \\ 2u + \quad\;\; 3w &= b_2 \\ 3u + v + 4w &= b_3 \end{aligned}$$

- We can solve this for $\mathbf{b} = (2, 5, 7)$ but not for $\mathbf{b} = (2, 5, 6)$

The column picture

- When will this fail?

$$
\begin{aligned}
u + v + w &= b_1 \\
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3u + v + 4w &= b_3
\end{aligned}
$$

- We can solve this for $\mathbf{b} = (2, 5, 7)$ but not for $\mathbf{b} = (2, 5, 6)$
- Why not?

The column picture

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \quad \begin{pmatrix}1\\0\\1\end{pmatrix} \quad \begin{pmatrix}1\\3\\4\end{pmatrix}$$

- LHS columns lie on a plane

The column picture

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times 2 \qquad \begin{pmatrix}1\\0\\1\end{pmatrix} \rightarrow -1 \qquad \begin{pmatrix}1\\3\\4\end{pmatrix} \times -2$$

- LHS columns lie on a plane
- $\rightarrow$ Every combination of them lies on that plane (that passes through origin)

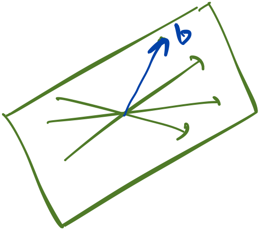
The column picture

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{pmatrix}1\\3\\4\end{pmatrix}$$

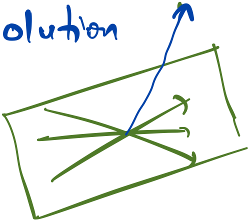
- LHS columns lie on a plane
- $\rightarrow$ Every combination of them lies on that plane (that passes through origin)
- If $\mathbf{b}$ does not lie on that plane $\rightarrow$

The column picture

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \qquad \begin{pmatrix}1\\0\\1\end{pmatrix} \qquad \begin{pmatrix}1\\3\\4\end{pmatrix}$$



- LHS columns lie on a plane
- $\rightarrow$ Every combination of them lies on that plane (that passes through origin)
- If **b** does not lie on that plane $\rightarrow$ No solution
- If **b** lies on that plane $\rightarrow$ Infinite solution



The column picture

- How do we know if the vectors lie on a plane?

The column picture

- How do we know if the vectors lie on a plane?
- One way: find a combination of them that gives zero

Summary

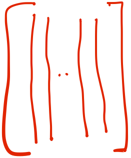
- Row Picture: Intersection of planes
- Column picture: Combination of columns

- If rows combine to give zero $\rightarrow$ columns would also do
- If n planes have no point in common (or infinite points), then the columns lie on the same plane!

Next

- Gaussian Elimination: An intuitive algorithm to solve the system of linear equations

Short/Fat matrix



Tall matrix

Rough work