

Determinants: Practice Problems

Moderate difficulty · 24 problems · Answer key at the end

1. Direct Evaluation

1. Evaluate $\begin{vmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & 5 & 1 \end{vmatrix}$ by cofactor expansion along a row or column of your choice.

2. Evaluate $\begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{vmatrix}$. (Hint: look at the block of zeros before you expand.)

3. Use row operations to reduce to triangular form, then evaluate:

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 3 \end{vmatrix}$$

Keep track of every row swap and scaling you perform.

4. Show that $\begin{vmatrix} x+1 & x & x \\ x & x+1 & x \\ x & x & x+1 \end{vmatrix} = 3x+1$ for every real x .

5. Evaluate $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$ by first adding all rows to the top row.

2. Factorisation and Symbolic Determinants

6. Prove the 3×3 Vandermonde identity

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b).$$

7. Factorise completely:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}.$$

8. Show that $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$, and state the condition on a, b, c under which this determinant vanishes.

9. Find all real x for which $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$, and state the multiplicity of each root.

10. Determine all values of k for which the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$ is singular.

3. Properties of Determinants

11. Let A be a 3×3 matrix with $\det A = 4$. Find

$$(a) \det(2A), \quad (b) \det(A^{-1}), \quad (c) \det(\operatorname{adj} A), \quad (d) \det(A^3 A^T).$$

12. Let A be a 4×4 matrix with $\det A = -3$. Find $\det(-2A)$, $\det(A^2)$ and $\det(\operatorname{adj} A)$.

13. For 3×3 matrices with $\det A = 2$ and $\det B = -5$, compute $\det(A^2 B^{-1} A^T)$.

14. Give an explicit 2×2 counterexample to the claim $\det(A + B) = \det A + \det B$. Then find a nonzero pair A, B for which the equation *does* hold.

15. Prove that if A is skew-symmetric ($A^T = -A$) and of odd order, then $\det A = 0$. Show by example that the conclusion fails for even order.

16. Prove that if A is orthogonal ($A^T A = I$), then $\det A = \pm 1$. Is the converse true?

17. Suppose A is 3×3 and idempotent ($A^2 = A$). What are the possible values of $\det A$? Exhibit a matrix realising each one.

18. Suppose $A^k = O$ for some positive integer k . Prove $\det A = 0$.

4. Adjugate, Inverse and Linear Systems

19. For $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$, compute $\det A$, the adjugate $\operatorname{adj} A$, and A^{-1} . Verify your answer by checking $AA^{-1} = I$.

20. Prove that $\det(\operatorname{adj} A) = (\det A)^{n-1}$ for an invertible $n \times n$ matrix A . Hence, if A is 3×3 and $\det(\operatorname{adj} A) = 25$, find $|\det A|$.

21. Solve by Cramer's rule:

$$2x - y + 3z = 9$$

$$x + y + z = 6$$

$$x - y + z = 2$$

22. Determine all values of k for which the homogeneous system

$$kx + y + z = 0, \quad x + ky + z = 0, \quad x + y + kz = 0$$

has a nontrivial solution, and describe the solution set in each case.

5. Challenge Problems

23. Let D_n be the determinant of the $n \times n$ tridiagonal matrix with 2 on the main diagonal and -1 on both adjacent diagonals, and all other entries 0. Show that $D_n = 2D_{n-1} - D_{n-2}$ for $n \geq 3$, and deduce that $D_n = n + 1$.
24. Let A be a 3×3 real matrix with $AA^T = I$ and $\det A = 1$. Prove that $\det(A - I) = 0$. Interpret the result geometrically.
25. Let A be an $n \times n$ matrix with integer entries. Prove that A^{-1} exists and has integer entries if and only if $\det A = \pm 1$.

Answer Key

1. 18.
2. 8. The matrix is block lower-triangular, so the determinant is $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \cdot \begin{vmatrix} 7 & 8 \\ 11 & 12 \end{vmatrix} = (-2)(-4) = 8$.
3. 1. One route: swap $R_1 \leftrightarrow R_2$ (sign -1), then clear the first column and continue; the triangular form has diagonal $1, 1, -1, 1$.
4. $3x + 1$. Add R_2 and R_3 to R_1 to get a common factor $(3x + 1)$ in the first row, then clear.
5. -3 . Each row sums to 3, so the row-sum trick gives a factor of 3; the remaining determinant is -1 .
6. $(b - a)(c - a)(c - b)$. Do $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$ and factor, or argue by the factor theorem (the determinant vanishes when any two of a, b, c coincide).
7. $(a - b)(b - c)(c - a)(a + b + c)$.
8. It equals $-(a^3 + b^3 + c^3 - 3abc) = -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$. It vanishes iff $a + b + c = 0$ or $a = b = c$.
9. The determinant factors as $(x - 1)^2(x + 2)$, so $x = 1$ (double) and $x = -2$ (simple).
10. The determinant is $(k - 1)(k - 2)$, so the matrix is singular exactly when $k = 1$ or $k = 2$ (the Vandermonde nodes $1, 2, k$ collide).
11. (a) $2^3 \cdot 4 = 32$. (b) $1/4$. (c) $(\det A)^2 = 16$. (d) $4^3 \cdot 4 = 256$.
12. $\det(-2A) = (-2)^4(-3) = -48$; $\det(A^2) = 9$; $\det(\operatorname{adj} A) = (-3)^3 = -27$.
13. $\frac{(2)^2 \cdot 2}{-5} = -\frac{8}{5}$.
14. Take $A = B = I_2$: $\det(A + B) = 4$ but $\det A + \det B = 2$. Equality holds for e.g. $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, where both sides equal 1. (Many other choices work.)
15. $\det A = \det(A^T) = \det(-A) = (-1)^n \det A$. For odd n this gives $2 \det A = 0$, so $\det A = 0$. For $n = 2$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is skew-symmetric with determinant 1.
16. $\det(A^T A) = (\det A)^2 = \det I = 1$, so $\det A = \pm 1$. The converse is false: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has determinant 1 but is not orthogonal.
17. From $A^2 = A$, $(\det A)^2 = \det A$, so $\det A \in \{0, 1\}$. The zero matrix gives 0 and I_3 gives 1; $\operatorname{diag}(1, 1, 0)$ also gives 0.

18. $(\det A)^k = \det(A^k) = 0$, and since $\det A$ is a scalar, $\det A = 0$.
19. $\det A = 1$, $\operatorname{adj} A = \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{pmatrix}$, $A^{-1} = \operatorname{adj} A$ (since $\det A = 1$).
20. From $A(\operatorname{adj} A) = (\det A)I_n$, taking determinants gives $\det A \cdot \det(\operatorname{adj} A) = (\det A)^n$, hence $\det(\operatorname{adj} A) = (\det A)^{n-1}$. For $n = 3$: $(\det A)^2 = 25$, so $|\det A| = 5$.
21. $D = -2$, $D_x = -2$, $D_y = -4$, $D_z = -6$, giving $x = 1$, $y = 2$, $z = 3$.
22. The coefficient determinant is $(k-1)^2(k+2)$, so nontrivial solutions occur iff $k = 1$ or $k = -2$. For $k = 1$ the system collapses to $x + y + z = 0$ (a plane of solutions, two free parameters). For $k = -2$ the solution set is the line $x = y = z$ (one free parameter).
23. Expanding along the first row gives $D_n = 2D_{n-1} - D_{n-2}$. With $D_1 = 2$ and $D_2 = 3$, the recurrence has repeated root 1, so $D_n = \alpha + \beta n$; the initial values force $D_n = n + 1$.
24. $\det(A - I) = \det(A - AA^T) = \det A \cdot \det(I - A^T) = \det(I - A) = (-1)^3 \det(A - I)$, so $\det(A - I) = 0$. Geometrically, A is a rotation of $\mathbb{R}^3$ and therefore fixes an axis: the eigenvalue 1 is Euler's rotation theorem.
25. ($\Leftarrow$) If $\det A = \pm 1$, then $A^{-1} = \frac{1}{\det A} \operatorname{adj} A = \pm \operatorname{adj} A$, whose entries are cofactors of A and hence integers. ($\Rightarrow$) If both A and A^{-1} are integer matrices, then $\det A$ and $\det(A^{-1})$ are integers whose product is 1, so $\det A = \pm 1$.