

Problem 1. Let $V(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} (x_j - x_i)$ denote the Vandermonde determinant. Prove that

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-2} & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} & x_2^n \\ \vdots & & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} & x_n^n \end{pmatrix} = \left(\sum_{i=1}^n x_i \right) V(x_1, \dots, x_n),$$

i.e. the matrix obtained from the Vandermonde matrix by skipping the power x^{n-1} .

Problem 2. Let T_n be the $n \times n$ tridiagonal matrix with a on the diagonal, b on the superdiagonal and c on the subdiagonal, and let $D_n = \det T_n$.

(a) Show $D_n = a D_{n-1} - bc D_{n-2}$ with $D_0 = 1$, $D_1 = a$.

(b) If $bc = 1$ and $a = 2 \cos \theta$ ($0 < \theta < \pi$), prove $D_n = \frac{\sin((n+1)\theta)}{\sin \theta}$.

(c) (Strang) Show that $a = 1$, $b = -1$, $c = 1$ gives the Fibonacci pattern $D_1, D_2, \dots = 1, 2, 3, 5, 8, \dots$, while $a = 1$, $b = c = 1$ gives the period-6 pattern $1, 0, -1, -1, 0, 1, \dots$

(d) Show the $-1, 2, -1$ matrix K_n has $\det K_n = n + 1$, and that its eigenvalues are $2 - 2 \cos \frac{k\pi}{n+1}$, $k = 1, \dots, n$.

Problem 3. (a) Prove the matrix determinant lemma: for invertible A and vectors u, v , $\det(A + uv^T) = \det A (1 + v^T A^{-1} u)$. (Hint: factor $\begin{pmatrix} I & 0 \\ v^T & 1 \end{pmatrix} \begin{pmatrix} I + uv^T & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ -v^T & 1 \end{pmatrix}$.)

(b) Compute $\det(D + \mathbf{1}\mathbf{1}^T)$ where $D = \text{diag}(a_1, \dots, a_n)$ with all $a_i \neq 0$.

(c) Compute the determinant of the $n \times n$ matrix with x on the diagonal and y everywhere else. For which x, y is it singular, and what is its rank then?

Problem 4. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $n \times n$ blocks.

(a) If A is invertible, prove $\det M = \det A \cdot \det(D - CA^{-1}B)$.

(b) If $AC = CA$, prove $\det M = \det(AD - CB)$ (even when A is singular).

(c) With $n = 2$, let $A = I$, $B = E_{12}$, $C = E_{21}$, $D = E_{11}$ (E_{ij} the matrix units). Compute $\det M$, and compare with $\det(AD - BC)$ and $\det A \det D - \det B \det C$. Explain why neither “natural guess” works.

Problem 5. (a) For $n \geq 3$, show that $\det[a_i + b_j]_{i,j=1}^n = 0$ and compute it for $n = 2$.

(b) Prove Cauchy’s formula

$$\det \left[\frac{1}{a_i + b_j} \right]_{i,j=1}^n = \frac{\prod_{i < j} (a_j - a_i)(b_j - b_i)}{\prod_{i,j} (a_i + b_j)}.$$

(c) Deduce that the 3×3 Hilbert matrix $H_{ij} = \frac{1}{i+j-1}$ has $\det H = \frac{1}{2160}$, and explain why H_n^{-1} has integer entries is plausible from (b) (no proof needed).

Problem 6. Let $C = \text{circ}(c_0, c_1, \dots, c_{n-1})$ be the circulant matrix whose first row is $(c_0, \dots, c_{n-1})$ and each subsequent row is the previous one shifted cyclically one place to the right.

- (a) Show that $\det C = \prod_{k=0}^{n-1} f(\omega^k)$ where $f(x) = \sum_j c_j x^j$ and $\omega = e^{2\pi i/n}$. (Hint: Fourier vectors $(1, \omega^k, \omega^{2k}, \dots)$.)
- (b) Prove $\det \text{circ}(1, 2, \dots, n) = (-1)^{n-1} \frac{(n+1)n^{n-1}}{2}$.
- (c) Evaluate $\det \text{circ}(1, 2, 3, 4)$.

Problem 7. Let A be real with $A^T = -A$.

- (a) Show $\det A = 0$ if n is odd.
- (b) For $n = 4$ with $a_{12} = a$, $a_{13} = b$, $a_{14} = c$, $a_{23} = d$, $a_{24} = e$, $a_{34} = f$, prove $\det A = (af - be + cd)^2$ (the Pfaffian squared).
- (c) Prove $\det(I + A) \geq 1$, with equality iff $A = 0$.
- (d) Show that if a real matrix satisfies $J^2 = -I$ then n is even and $\det J = 1$.

Problem 8. Let A be $n \times n$ with $n \geq 2$.

- (a) Prove $\det(\text{adj } A) = (\det A)^{n-1}$ for every A .
- (b) Prove $\text{adj}(\text{adj } A) = (\det A)^{n-2} A$.
- (c) Show $\text{rank}(\text{adj } A) = n, 1, 0$ according as $\text{rank } A = n, n-1, \leq n-2$.
- (d) If A has integer entries and A^{-1} also has integer entries, prove $\det A = \pm 1$. Conversely, why does $\det A = \pm 1$ guarantee an integer inverse?

Problem 9. (a) Show $\det[\min(i, j)]_{n \times n} = 1$ by writing it as LL^T .

- (b) Show $\det[\max(i, j)]_{n \times n} = (-1)^{n-1} n$.
- (c) (Smith, 1876) Prove $\det[\gcd(i, j)]_{n \times n} = \prod_{k=1}^n \varphi(k)$, where φ is Euler's totient. (Hint: $\gcd(i, j) = \sum_{d|i, d|j} \varphi(d)$.) Evaluate for $n = 6$.

Problem 10. (a) Prove Jacobi's formula $\frac{d}{dt} \det A(t) = \text{tr}(\text{adj } A(t) A'(t))$, and hence $\frac{d}{dt} \det A = \det A \text{tr}(A^{-1} A')$ when A is invertible.

- (b) Prove $\det(e^A) = e^{\text{tr } A}$. Deduce that e^A is always invertible, and that $e^A \in SO(n)$ when A is skew-symmetric.
- (c) (Liouville / Abel) If $X(t)$ is a fundamental matrix of $x' = P(t)x$, show the Wronskian $W = \det X$ satisfies $W(t) = W(t_0) \exp \int_{t_0}^t \text{tr } P(s) ds$.

Problem 11. (a) Using $A = QR$ (Gram-Schmidt), prove $|\det A| \leq \prod_{j=1}^n \|a_j\|$ where a_j are the columns of A . When does equality hold?

- (b) Show every $n \times n$ matrix with entries ± 1 has $|\det A| \leq n^{n/2}$, and that $\det A$ is divisible by 2^{n-1} .

- (c) Hence prove the maximum determinant of a 3×3 ± 1 matrix is exactly 4 (exhibit one), and find a 4×4 ± 1 matrix with determinant 16.

Problem 12. (a) State the Cauchy–Binet formula for $\det(AB)$ when A is $k \times n$ and B is $n \times k$ with $k \leq n$. Use it to prove Lagrange’s identity

$$\left(\sum a_i^2\right)\left(\sum b_i^2\right) - \left(\sum a_i b_i\right)^2 = \sum_{i < j} (a_i b_j - a_j b_i)^2.$$

- (b) Show that the k -dimensional volume of the parallelepiped spanned by the columns of an $n \times k$ matrix M is $\sqrt{\det(M^T M)}$.

- (c) For $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, compute $\det(AA^T)$ two ways, and find the area of the parallelogram in $\mathbb{R}^3$ spanned by the rows of A .

Problem 13. The Laplacian of a graph G on n vertices is $L = D - \mathcal{A}$ (degree matrix minus adjacency matrix). Kirchhoff’s Matrix–Tree Theorem says the number of spanning trees $\tau(G)$ equals the determinant of L with any one row and the corresponding column deleted.

- (a) Explain why $\det L = 0$ always.
(b) Prove Cayley’s formula $\tau(K_n) = n^{n-2}$.
(c) Prove $\tau(C_n) = n$ for the cycle, using Problem 2(d).
(d) Compute $\tau(K_{2,3})$ via a 4×4 determinant.

Problem 14. For $p(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$ define the companion matrix

$$C_p = \begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix}.$$

- (a) Prove $\det(xI - C_p) = p(x)$.
(b) Deduce $\det C_p = (-1)^n c_0$, and that every monic polynomial is a characteristic polynomial.
(c) The Fibonacci recurrence $F_{k+2} = F_{k+1} + F_k$ can be written $u_{k+1} = Fu_k$ with $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Using $\det(F^n)$ prove Cassini’s identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

Problem 15. (a) Show $\det : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is the unique function that is linear in each row, alternating, and equals 1 at I ; deduce $\det(AB) = \det A \det B$ by considering $A \mapsto \det(AB)/\det A$.

- (b) Among all $n \times n$ real matrices whose rows have length 1, find $\max |\det A|$ and characterize the maximizers. Among matrices with $\|A\|_F^2 = \sum a_{ij}^2 = n$, find $\max \det A$.
(c) Let A be symmetric positive definite. Prove $\det A \leq \prod_i a_{ii}$, and that $(\det A)^{1/n} \leq \frac{1}{n} \operatorname{tr} A$ (AM–GM on eigenvalues). When is $\det(A + B) \geq \det A + \det B$ for SPD A, B ?