

Matrix Theory

Practice Questions

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Part I Linear Systems, Elimination and $A = LU$

1. True or false: a system of linear equations with real coefficients (over $\mathbb{R}$) can have exactly three solutions. Justify your answer.
2. True or false: if a 3×3 system has no solution, then two of its three planes must be parallel. If false, describe the geometry of a counterexample.
3. From slides: “if the rows combine to give zero, the columns also do”. (That is: dependent rows imply dependent columns.) Explain why this holds for square matrices. Does it hold for every 2×3 matrix? Can it fail for a 3×2 matrix?
4. Explain, using the column picture, why no sequence of row exchanges can turn a singular matrix into one with a full set of pivots.
5. Let A be $n \times n$. Prove that if $Ax = b$ has at least one solution for *every* b , then it has exactly one solution for every b .
6. True or false: if $AB = 0$, then $A = 0$ or $B = 0$. For an $m \times n$ matrix A , what extra condition on A would make “ $AB = AC \Rightarrow B = C$ ” valid? What does this condition say when A is square?
7. Assume elimination needs no row exchanges. Why do we record elimination as $A = LU$ rather than $EA = U$? Without computing, explain why $E = E_{32}E_{31}E_{21}$ mixes the multipliers while $L = E^{-1}$ holds each multiplier ℓ_{ij} exactly in position (i, j) .
8. When is $PA = LU$ needed instead of $A = LU$? Why can all the row exchanges be collected into a single permutation P applied before elimination?
9. Prove that for square matrices $AB = I$ implies $BA = I$. Show by example that this fails for rectangular matrices.
10. Factor $A = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix}$ as $A = LU$, and use the factors (forward, then back substitution) to solve $Ax = (3, 14)$.

Part II Vector Spaces, Subspaces and $Ax = b$

11. Decide which of these are subspaces of $\mathbb{R}^2$, giving a reason each time: (a) a line not through the origin; (b) the closed first quadrant $\{x \geq 0, y \geq 0\}$; (c) the union of the closed first and third quadrants; (d) a line through the origin.

12. True or false, with reasons: (a) row operations always preserve the null space; (b) row operations always preserve the column space.
13. Why do the 2×2 matrices with trace 0 form a subspace of $M_{2 \times 2}$, while those with determinant 0 do not?
14. Give a 2×2 matrix with $C(A) = N(A)$ and describe what it does to vectors. Then explain why no 3×3 matrix can have $C(A) = N(A)$.
15. If $EA = U$ with E invertible, why is $N(A) = N(U)$ while $C(A)$ and $C(U)$ can differ? What exact relation holds between $C(A)$ and $C(U)$, and which columns of A form a basis of $C(A)$?
16. Prove that the union $U \cup W$ of two subspaces is a subspace if and only if $U \subseteq W$ or $W \subseteq U$.
17. Explain why $x_p + x_n$ (with $Ax_p = b$ and $x_n \in N(A)$) is always a solution, and why *every* solution of $Ax = b$ has this form.
18. The complete solution of $Ax = b$ is $x = (1, 2, 0) + c(1, 1, 1)$. What can you say about n , r and $N(A)$? Could A be 2×3 ? 3×3 ? 5×3 ? Can $Ax = b'$ have exactly one solution for some other b' ?
19. For each of the four cases $r = m = n$, $r = n < m$, $r = m < n$, and $r < m$, $r < n$, describe the shape of $R = \text{rref}(A)$ and the possible numbers of solutions of $Ax = b$.
20. Prove the equivalences: $Ax = b$ is solvable for every $b \iff r = m \iff A$ has a right inverse C (with $AC = I_m$).
21. Find the complete solution, in the form $x_p + x_n$, of $x_1 + 2x_2 + x_3 = 4$, $2x_1 + 4x_2 + 3x_3 = 9$.

Part III Independence, Basis, Dimension and the Four Subspaces

22. True or false: three vectors are linearly dependent if and only if one of them is a multiple of another.
23. Why are the coordinates of a vector with respect to a basis unique? Which property of a basis gives *existence* of coordinates, and which gives *uniqueness*?
24. Suppose v_1, v_2, v_3 are independent. Without computing, decide whether (a) $v_1, v_1 + v_2, v_1 + v_2 + v_3$ and (b) $v_1 - v_2, v_2 - v_3, v_3 - v_1$ are independent.
25. From slides: “the vectors in a basis are not intrinsic to the space, but the number of vectors in every basis is”. Explain what this means and why it is true.
26. Two planes through the origin in $\mathbb{R}^3$: what can the dimension of their intersection be? What does this imply about whether two such planes can be orthogonal subspaces?
27. Prove: if $v_1, \dots, v_m$ span V , then any $n > m$ vectors in V are dependent. Deduce that all bases of a finite-dimensional space have the same size.
28. True or false, with reasons: (a) $N(A)$ and $N(A^T)$ always have the same dimension; (b) the row space of A always equals the column space of A .
29. Why do the nonzero rows of U give a basis for the row space of A , while the pivot columns of U do *not* give a basis for $C(A)$?
30. Can a matrix have a row space containing $(1, 1, 1)$ and a null space containing $(1, 0, -1)$? Containing $(1, 0, 0)$? Explain using the relationship between these two subspaces.

31. For a real matrix A , explain why $N(A^T A) = N(A)$, and conclude that $\text{rank}(A^T A) = \text{rank } A$. Why does this matter for least squares?
32. Prove the rank-nullity theorem, $\dim C(A) + \dim N(A) = n$, by constructing bases (without counting pivots).
33. Find bases for all four fundamental subspaces of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \end{bmatrix}$ and check that the dimensions are $r, n - r, r, m - r$.

Part IV Linear Transformations

34. State the two properties that define a linear transformation and explain why “ $T(0) = 0$ ” is a quick test. Is $T(x, y) = (x + 1, y)$ linear? Is $T(x, y) = (x^2, y)$?
35. Why is a linear transformation completely determined by what it does to a basis? In the standard matrix of T , where do the images $T(e_1), \dots, T(e_n)$ appear?
36. For the orthogonal projection P onto a line through the origin in $\mathbb{R}^2$ and the reflection H across the same line: describe the kernel and range of P ; explain geometrically why $P^2 = P$ and $H^2 = I$; and explain why $H = 2P - I$.
37. Why is differentiation $D : P_3 \rightarrow P_3$ not invertible? Why is “integrate from 0, then differentiate” the identity on P_2 , while “differentiate, then integrate” is not the identity on P_3 ?
38. Let $T : V \rightarrow V$ be a linear operator on a finite-dimensional space. Its matrix depends on the basis chosen for V . What changes, and what stays the same? Why might one look for a basis in which the matrix is diagonal? Give such a basis for the orthogonal projection onto a line in $\mathbb{R}^2$.
39. Prove that a linear map T is one-to-one iff $\ker T = \{0\}$, and that for $T : V \rightarrow V$ with V finite-dimensional, one-to-one $\iff$ onto. Show that the last equivalence fails in infinite dimensions.

Part V Orthogonality, Projections, Least Squares and QR

40. What is the difference between two subspaces being *orthogonal* and being *orthogonal complements*? Do the floor and a wall of a classroom (thought of as planes through a common corner taken as the origin) form orthogonal subspaces?
41. True or false, with reasons: (a) if $x \perp y$ and $y \perp z$, then $x \perp z$; (b) mutually orthogonal nonzero vectors are independent, and conversely independent vectors are orthogonal.
42. Explain this statement from slides: “from the row space to the column space, the matrix A is invertible”. Why does each $b \in C(A)$ come from exactly one x_r in the row space, although it may come from many x ?
43. Among all solutions of a consistent system $Ax = b$, why is the one lying in the row space the shortest?
44. Explain how to test whether $b \in C(A)$ using a basis of $N(A^T)$, without doing elimination on b . Why does this test work?
45. Prove the Fredholm alternative: exactly one of the following holds: (a) $Ax = b$ has a solution; (b) there is a y with $A^T y = 0$ and $y^T b \neq 0$.

46. For the projection of b onto the line through a nonzero vector a : why must the error $e = b - p$ be perpendicular to a ? What is the rank of $P = \frac{aa^T}{a^T a}$, and why is $P^2 = P$?
47. The matrix $M = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ satisfies $M^2 = M$. Is it the orthogonal projection onto a line? Describe what it does, and explain why symmetry of a projection matrix matters.
48. The correlation coefficient of two data sets (each with nonzero variance) is the cosine of the angle between their mean-centred data vectors. Why does $|r| \leq 1$ follow immediately, and what does $|r| = 1$ mean?
49. Prove: if $M^2 = M$ and $\|Mx\| \leq \|x\|$ for every x , then $N(M) \perp C(M)$, so M is an orthogonal projection.
50. Project $b = (1, 2, 3)$ onto the line through $a = (1, 1, 1)$. Give $\hat{x}$, p , e , and the distance from b to the line.
51. True or false, for the least-squares line $b = C + Dt$: (a) it always passes through at least one data point; (b) it passes through $(\bar{t}, \bar{b})$; (c) its residuals sum to zero.
52. Explain the geometric derivation of the normal equations: why does “the closest point of $C(A)$ to b ” lead to $A^T(b - A\hat{x}) = 0$? Why are these called the *normal* equations?
53. Why can adding a column (a new feature) to A never increase the minimum least-squares error? Show that a polynomial of degree at most $m - 1$ can interpolate m observations (t_i, b_i) with pairwise distinct t_i exactly. Why might such a model overfit noisy data?
54. In fitting $b = C + Dt$ the errors are measured vertically. Why? Why do “regress b on t ” and “regress t on b ” generally give different lines?
55. Suppose the data are $b = Ax_{\text{true}} + n$, where n is noise and A has independent columns. Show that $\hat{x} - x_{\text{true}} = (A^T A)^{-1} A^T n$, and conclude exactly when least squares recovers x_{true} .
56. Fit $b = C + Dt$ to the data $(0, 1), (1, 2), (2, 4)$, and verify that the error vector is orthogonal to both columns of A .
57. True or false: (a) an orthogonal matrix is any square matrix with orthogonal columns; (b) if Q is $m \times n$ ($m > n$) with orthonormal columns, then $QQ^T = I$.
58. Let A be $m \times n$ with $\text{rank } A = n$, and let $A = QR$ be its reduced QR factorization ($Q^T Q = I_n$, R square and invertible). How does this simplify least squares? Show that the normal equations become $R\hat{x} = Q^T b$, and explain why this route is numerically preferable to forming $A^T A$.
59. Show that a matrix which is both orthogonal and upper triangular must be diagonal with entries ± 1 . Use this to prove that the reduced QR factorization of a full-column-rank matrix ($Q^T Q = I_n$, R upper triangular with positive diagonal) is unique.
60. Apply Gram–Schmidt to the columns of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$ to obtain $A = QR$.