Deep Learning

06 Backpropagation-2

Dr. Konda Reddy Mopuri
Dept. of Artificial Intelligence
IIT Hyderabad
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Consider a specific Layer

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- Then wrt the parameters

$$\frac{\partial \ell}{\partial w_{i,j}^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}} x_j^{(l-1)} \text{ and } \frac{\partial \ell}{\partial b_i^{(l)}} = \frac{\partial \ell}{\partial s_i^{(l)}}$$

Jocobian in Tensorial form

- $\psi : \mathcal{R}^N \rightarrow \mathcal{R}^M$ then $\left[\frac{\partial \psi}{\partial x}\right] = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \cdots & \frac{\partial \psi_1}{\partial x_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi_M}{\partial x_1} & \cdots & \frac{\partial \psi_M}{\partial x_N} \end{bmatrix}$

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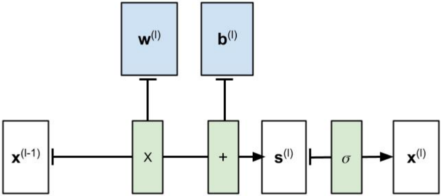
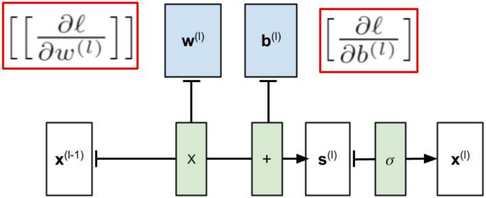
- $\psi : \mathcal{R}^{N \times M} \rightarrow \mathcal{R}$ then $\left[\left[\frac{\partial \psi}{\partial x}\right]\right] = \begin{bmatrix} \frac{\partial \psi}{\partial w_{1,1}} & \cdots & \frac{\partial \psi}{\partial w_{1,M}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \psi}{\partial w_{N,1}} & \cdots & \frac{\partial \psi}{\partial w_{N,M}} \end{bmatrix}$

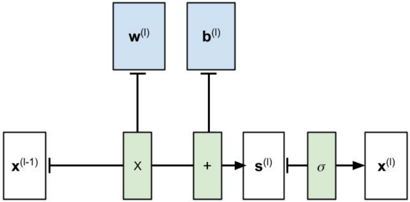
Forward Pass



Goal of Backward Pass



Begin from succeeding layer



$$\frac{\partial \ell}{\partial x^{(l)}}$$

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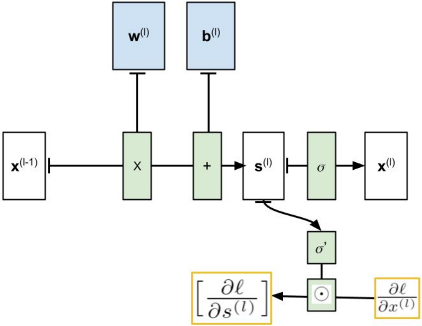

Begin from succeeding layer

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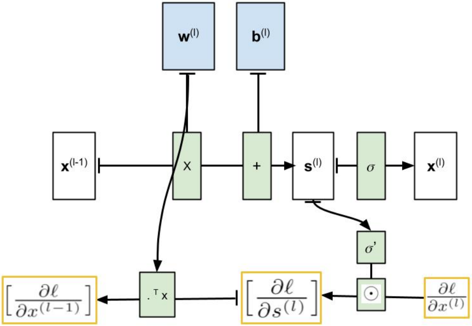

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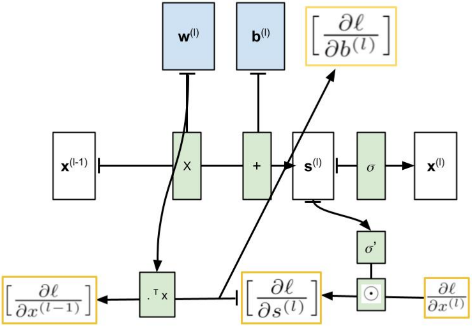

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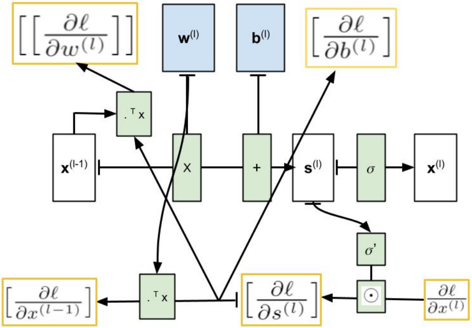

Update the parameters

- $W^{(l)} = W^{(l)} - \eta \left[\left[\frac{\partial \ell}{\partial w^{(l)}}\right]\right]$ and $\mathbf{b}^{(l)} = \mathbf{b}^{(l)} - \eta \left[\frac{\partial \ell}{\partial b^{(l)}}\right]$

Observations

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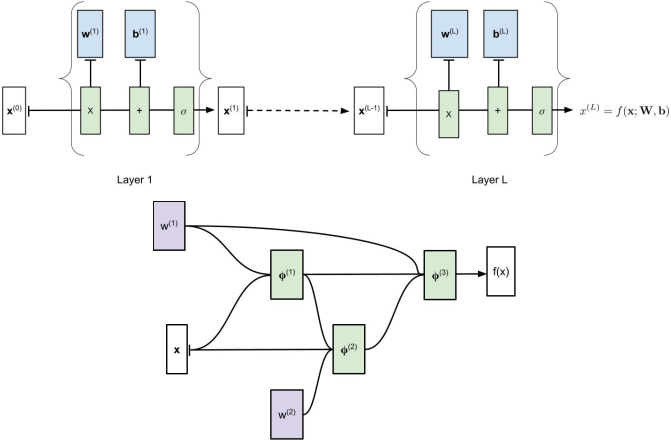
- BP is basically simple: applying chain rule iteratively
- It can be expressed in tensorial form (similar to the forward pass)
- Heavy computations are with the linear operations
- Nonlinearities go into simple element wise operations
- BP Needs all the intermediate layer results to be in memory
- Takes twice the computations of forward pass

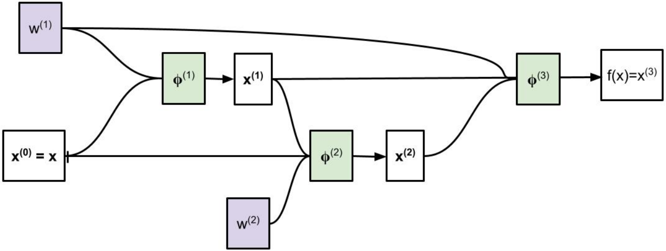
Beyond MLP

- We can generalize MLP



To an arbitrary Directed Acyclic Graph (DAG)

Forward pass in the computational graph



- $x^{(0)} = x$

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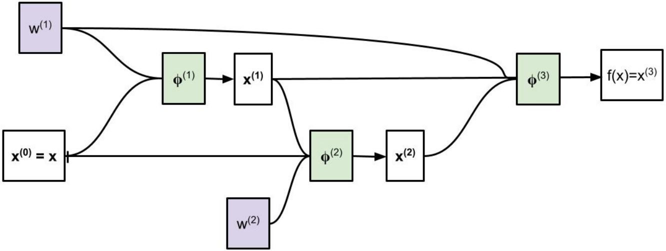

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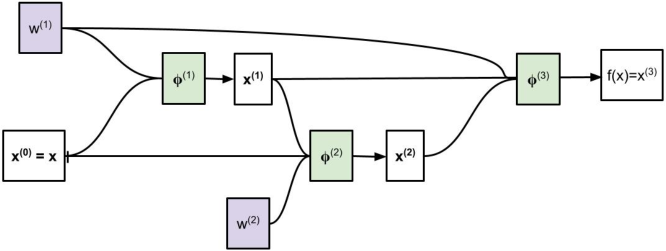

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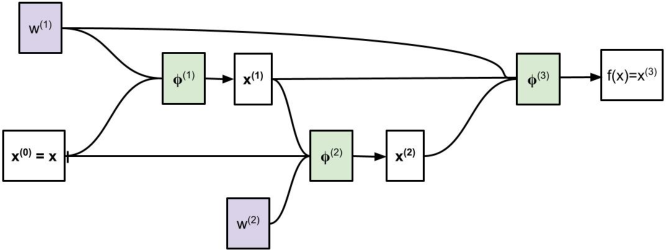

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- $x^{(1)} = \phi^{(1)}(x^{(0)}; w^{(1)})$
- $x^{(2)} = \phi^{(2)}(x^{(0)}, x^{(1)}; w^{(2)})$
- $f(x) = x^{(3)} = \phi^{(3)}(x^{(1)}, x^{(2)}; w^{(1)})$

Notation: Jacobian of a general transformation

- $$\text{if } (a_1 \dots a_Q) = \phi(b_1 \dots b_R) \text{ then we use the notation} \tag{3}$$

$$\left[\frac{\partial a}{\partial b}\right] = J_\phi^T = \begin{bmatrix} \frac{\partial a_1}{\partial b_1} & \cdots & \frac{\partial a_Q}{\partial b_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial a_1}{\partial b_R} & \cdots & \frac{\partial a_Q}{\partial b_R} \end{bmatrix} \tag{4}$$

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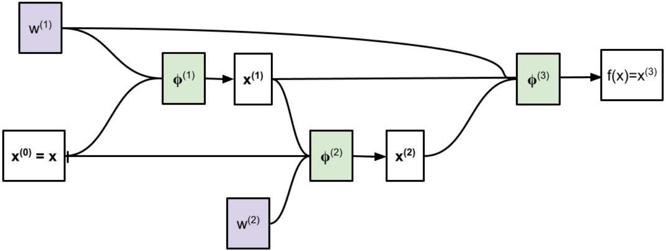
Backward pass



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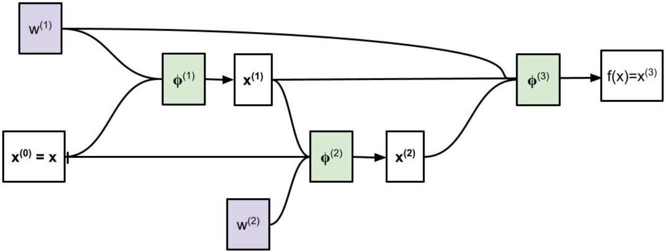

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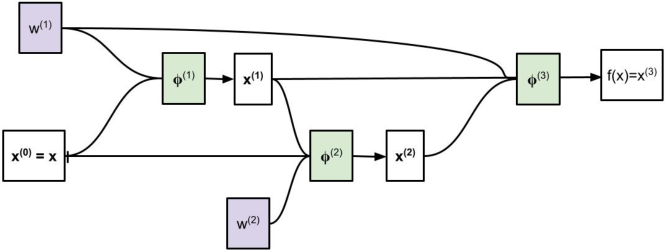

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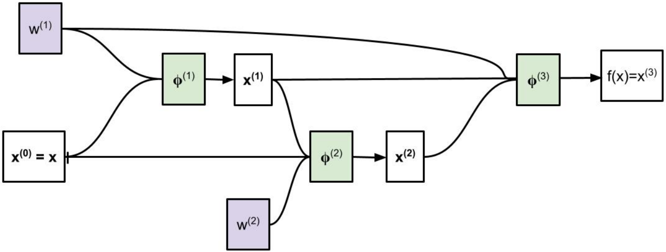

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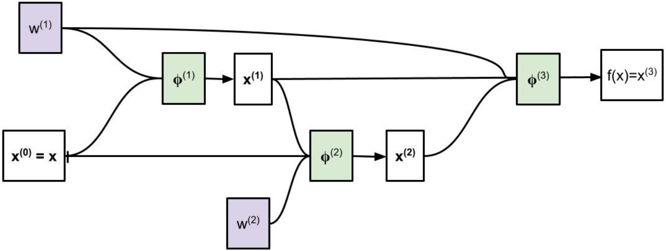

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- Remember, our loss function is only a proxy for the classification error
- Minimizing the proxy may not minimize the actual
- i.e., ideal function (separation for classification) may not be a feasible optimum for the chosen loss function

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- Active research!