Deep Learning

18 Variational Autoencoder

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Autoencdoers

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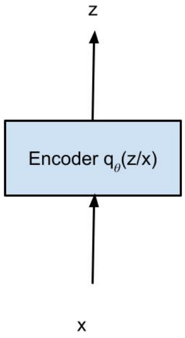
2. We attempted to project the data into the latent space and model it via a probability distribution
3. This wasn't satisfying

Variational Autoencoders

1. 'Regularized' autoencoder to enforce latent space 'organization'

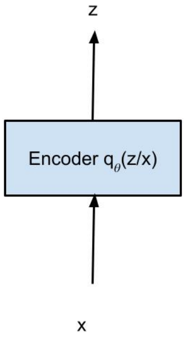
Variational Autoencoders

1. Key idea is to make both Encoder and Decoder stochastic
   - instead of encoding an i/p as a single point, we encode it as a distribution over the latent space



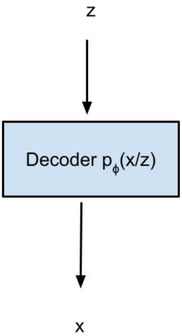
Variational Autoencoders

1. Key idea is to make both Encoder and Decoder stochastic
   - instead of encoding an i/p as a single point, we encode it as a distribution over the latent space
2. Latent variable z is drawn from a probability distribution for the given input x



Variational Autoencoders

1. Then, the reconstruction is chosen probabilistically from the sampled z



VAE Encoder

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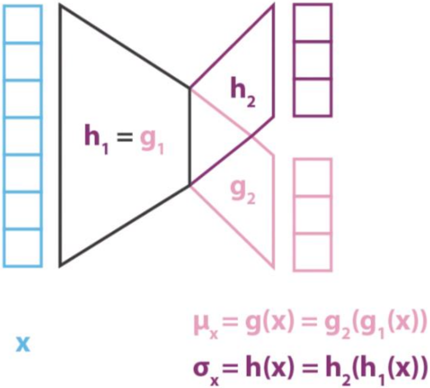
3. NN implementation of the encoder gives (for every input x) a vector mean and a diagonal covariance

VAE Encoder



VAE Decoder

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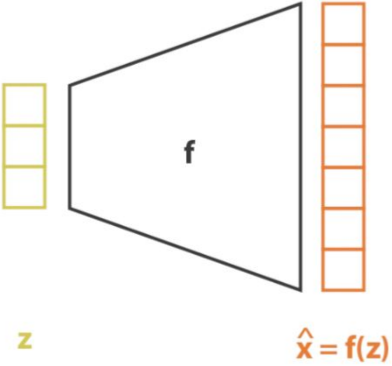
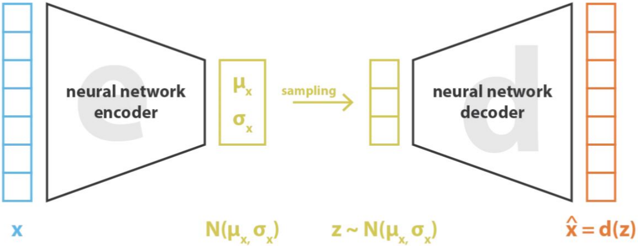
3. Reconstruction of x is via sampling (with some assumptions, the data sample can be output)

VAE Decoder



VAE Forward pass



VAE loss function

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1. Loss for AE: l_2 distance between the input and its reconstruction
2. In case of VAE: we need to learn parameters of two probability distributions
3. For each input x_i we maximize expected value of returning x_i (or, minimize the NLL)

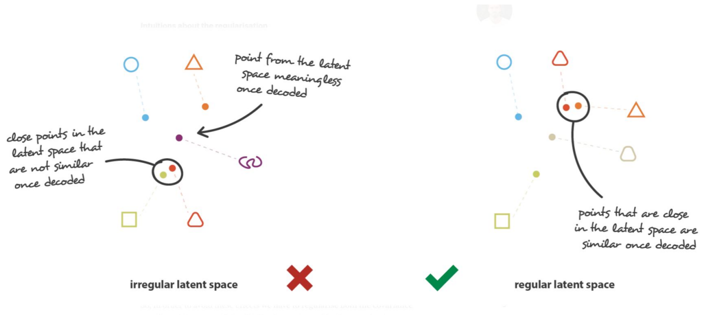
$$-\mathbb{E}_{z \sim q_\theta(z/x_i)}[log\, p_\phi(x_i/z)]$$

VAE loss function

$$-\mathbb{E}_{z \sim q_\theta(z/x_i)}[log\, p_\phi(x_i/z)]$$

1. Problem: Input images may be memorized in the latent space
   - $\rightarrow$ similar inputs may get different representations in z space
   - $\rightarrow$ close points in the latent space should not give two completely different contents once decoded

VAE loss function



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1. Continuity and Completeness: We prefer continuous latent representations to give meaningful parameterization (e.g. smooth transition between i/ps)
2. Solution: Force $q_\theta(z/x_i)$ to be close to a standard distribution (e.g. Gaussian)

VAE loss function

$$l_i(\theta, \phi) = -\mathbb{E}_{z \sim q_\theta(z/x_i)}[log\, p_\phi(x_i/z)] + \mathbb{KL}(q_\theta(z/x_i)||p(z))$$

1. First term promotes recovery, sencond term keeps encoding continuous (beats memorization)

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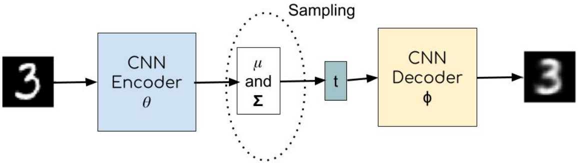
1. Problem: Differentiating over θ and ϕ



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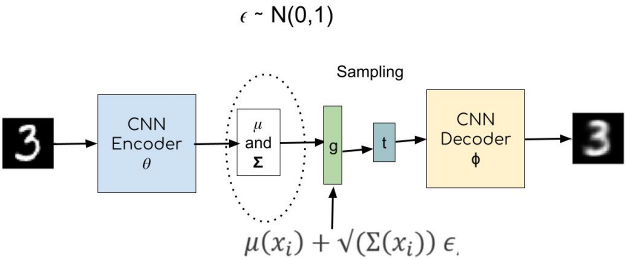
1. Reparameterization: Draw samples from N(0,1) $\rightarrow$ doesn't depend on parameters



Generation with VAE

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- For simplicity, in practice, only the means of the pixels are inferred (deterministic)

Generation with VAE

Figure credits: Wojceich

Generation with VAE

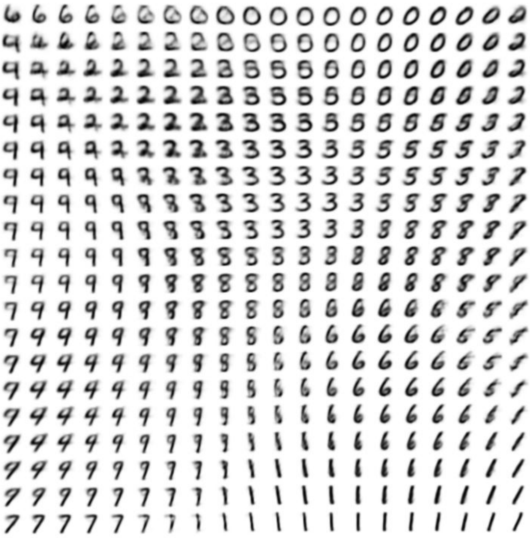

Figure credits: Kingma et al.

Edit/Manipulate samples with VAE

The Evidence Lower Bound (ELBO)

Latent Variable Models

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   - account for measurement errors
   - controlled/customized generation of the samples

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Latent Variable Models

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2. Because the latent variables explain the data in a simpler way

Latent Variable Models - terminology

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3. $p(z)$ prior distribution that models the behavior of latent variables

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4. $p(z/x)$, posterior, describes the latent variables that can be produced by a data sample

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3. Represented by the likelihood $p(x/z)$

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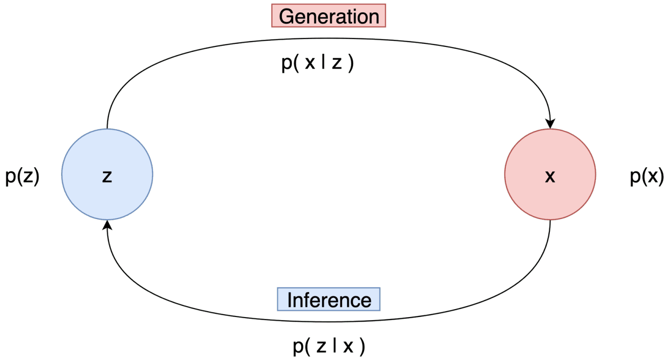
2. Formulated by the posterior distribution $p(z/x)$

Generation-Inference

- If we assume that we (somehow) know the likelihood $p(x/z)$, the posterior $p(z/x)$, the marginal $p(x)$, and the prior $p(z)$



Aim/Question of latent Variable Models

1. How to find these distributions?

These can be connected

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5. Variational inference suggests to use another (known) distribution $(q_\phi(z/x))$ to approximate the posterior $\rightarrow$ (allows to compute the evidence and sample)

Variational Inference

1. $p_\theta(z/x) \approx q_\phi(z/x)$

Variational Inference

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4. KL Divergence

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5. $D_{KL} = \mathbb{E}_{q_\phi}[log\, q_\phi(z/x)] - \mathbb{E}_{q_\phi}[log\, p_\theta(z,x)] + \mathbb{E}_{q_\phi}[log\, p_\theta(x)]$

ELBO

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4. We can't compute because we don't have its analytical form

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5. This is the lower bound on the evidence
6. Now, in order to reduce the D_{KL}, we can maximize the ELBO

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5. These represent the reconstruction and KLD (approx. posterior, the prior)

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2. ELBO $= -\mathbb{E}_{q_\phi}[log\, q_\phi(z/x)] + \mathbb{E}_{q_\phi}[log\, p_\theta(x/z)] + \mathbb{E}_{q_\phi}[log\, p_\theta(z)]$
3. ELBO $= \mathbb{E}_{q_\phi}[log\, p_\theta(x/z)] - \mathbb{E}_{q_\phi}[log\, q_\phi(z/x)] + \mathbb{E}_{q_\phi}[log\, p_\theta(z)]$
4. ELBO $= \mathbb{E}_{q_\phi}[log\, p_\theta(x/z)] - \mathbb{E}_{q_\phi}\left[log\, \frac{q_\phi(z/x)}{p_\theta(z)}\right]$
5. These represent the reconstruction and KLD (approx. posterior, the prior)
6. VAEs model $p_\theta(x/z)$ and $q_\phi(z/x)$ as neural networks