# Deep Learning

## 03 MLP: Representation Power of an MLP

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# So far

1. Any Boolean function of $n$ inputs can be exactly represented with one hidden layer!

# Universal Approximation (for real functions)



1. We can represent any continuous function $(f : \mathcal{R}^m \rightarrow \mathcal{R}^n)$ to any desired approximation $(|g(x) - f(x)| < \epsilon)$ with a linear combination of sigmoid neurons

# Universal Approximation (for real functions)

1. We can represent any continuous function $(f : \mathcal{R}^m \rightarrow \mathcal{R}^n)$ to any desired approximation $(|g(x) - f(x)| < \epsilon)$ with a linear combination of sigmoid neurons
2. In other words, neural networks with a single hidden layer can be used to approximate any continuous function to any desired precision

# Universal Approximation



Math. Control Signals Systems (1989) 2: 303–314

Mathematics of Control, Signals, and Systems



**Approximation by Superpositions of a Sigmoidal Function***

G. Cybenko†

*Neural Networks*, Vol. 4, pp. 251–257, 1991


0893-6080/91 $3.00 + .00


*ORIGINAL CONTRIBUTION*

**Approximation Capabilities of Multilayer Feedforward Networks**

KURT HORNIK

Technische Universität Wien, Vienna, Austria

# Universal Approximation



1. Let's look at the visual proof!

# Universality with 1-input and 1-output



- Two hidden units and one output unit

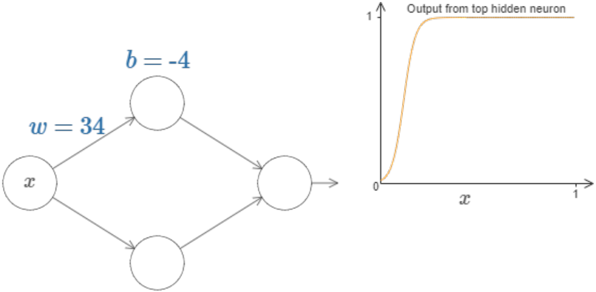
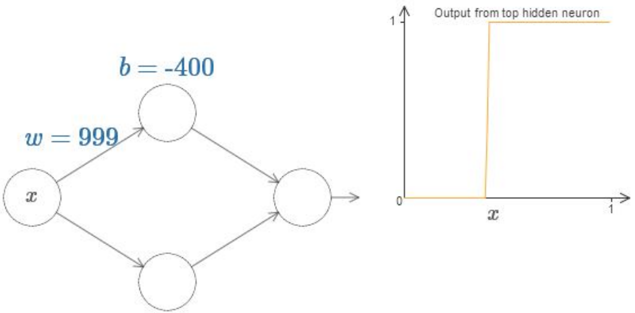



Figure from Michael Nielsen's NNDL textbook

# Universality with 1-input and 1-output

- Sigmoid neurons can closely approximate a step function!

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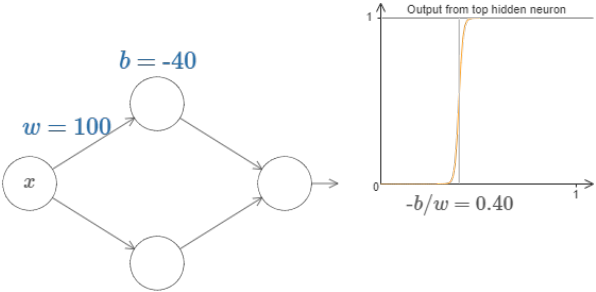
# Universality with 1-input and 1-output

- Let's simplify the neuron representation with a single parameter (s)

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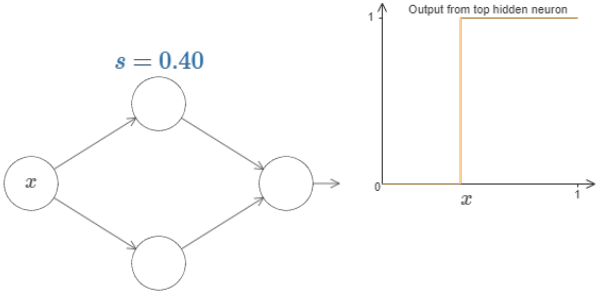
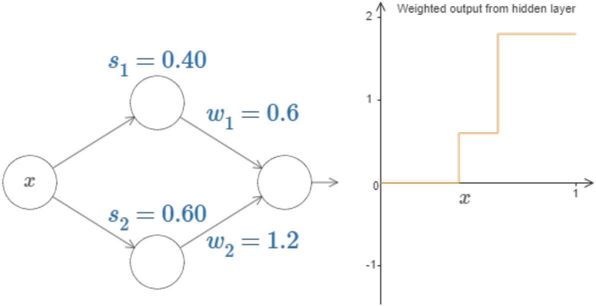



Figure from Michael Nielsen's NNDL textbook

# Universality with 1-input and 1-output

- Weighted output of hidden neurons

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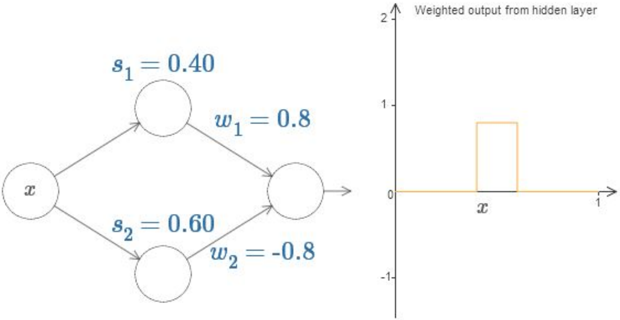
# Universality with 1-input and 1-output

- Can output a pulse/tower of desired width and height!

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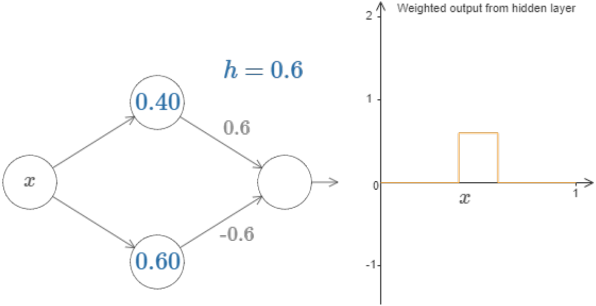


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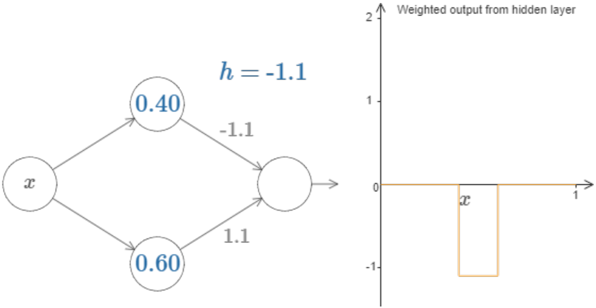


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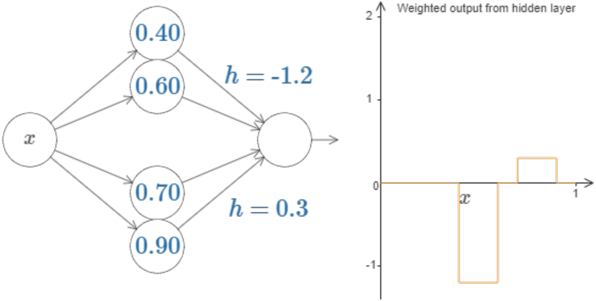
# Universality with 1-input and 1-output



- With more neurons in the hidden layer, more towers!

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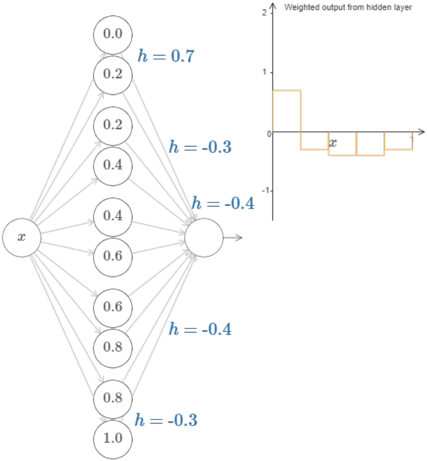


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# Universality with 1-input and 1-output

1. Note that we computed only the weighted sum of the hidden outputs

# Universality with 1-input and 1-output

1. Note that we computed only the weighted sum of the hidden outputs
2. It's not the output of our MLP

# Universality with 1-input and 1-output



- For approximating $f(x)$, the input to the output neuron has to be $\sigma^{-1}(f(x))$ (note that the bias is zero)

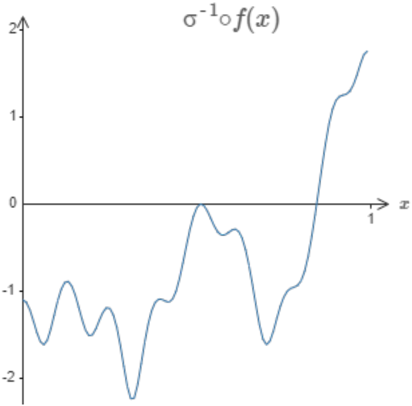
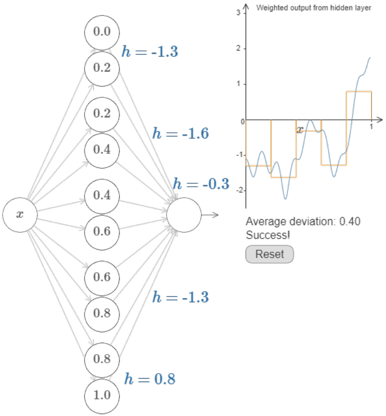



Figure from Michael Nielsen's NNDL textbook

# Universality with 1-input and 1-output

- Manipulating the width and height of the towers → a better approximation of the function

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# Universality with multiple inputs

- Let's consider two input variables

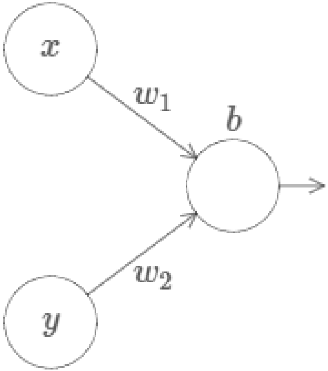
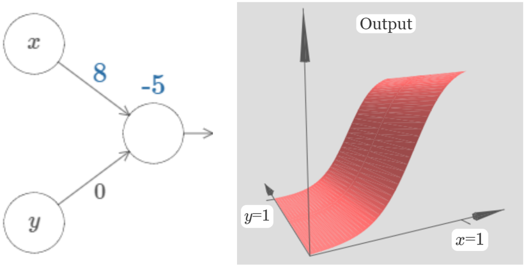



Figure from Michael Nielsen's NNDL textbook

# Universality with multiple inputs

- Let's set $w_2 = 0$

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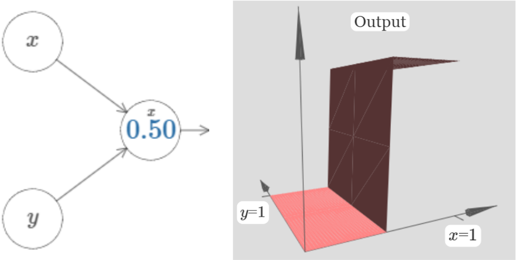
# Universality with multiple inputs

- As seen earlier, let's approximate the step function

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- As seen earlier, let's approximate the step function
- Use a single parameter $s = -b/w$ to represent

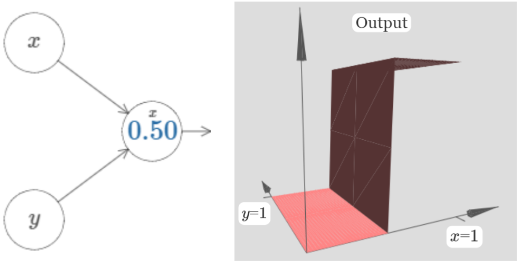
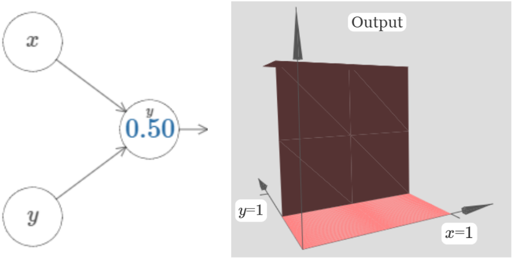



Figure from Michael Nielsen's NNDL textbook

# Universality with multiple inputs



- The step function in the $y$ direction

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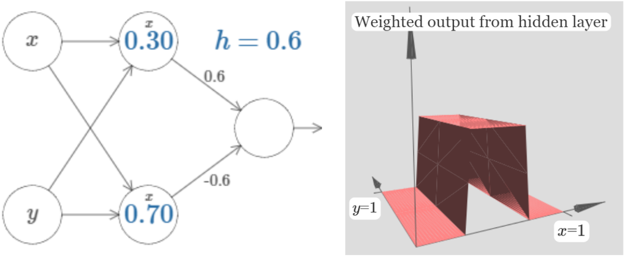
# Universality with multiple inputs

- Towards the tower in 3D

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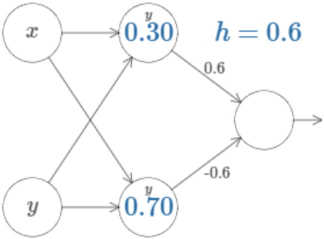


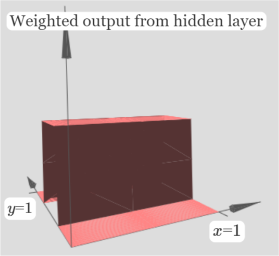


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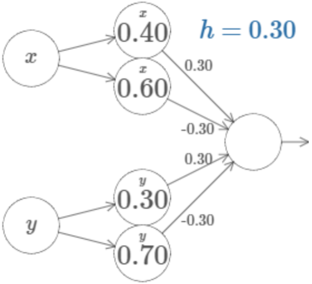


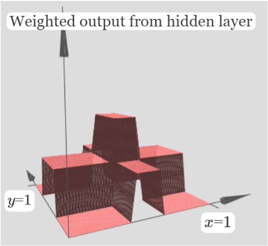


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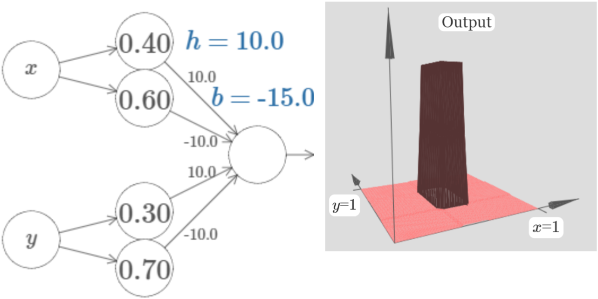


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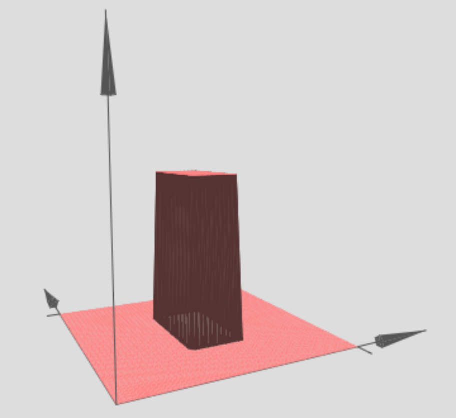

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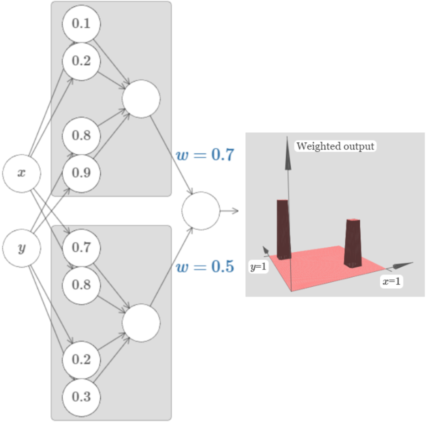
- Several of the towers can approximate arbitrary functions

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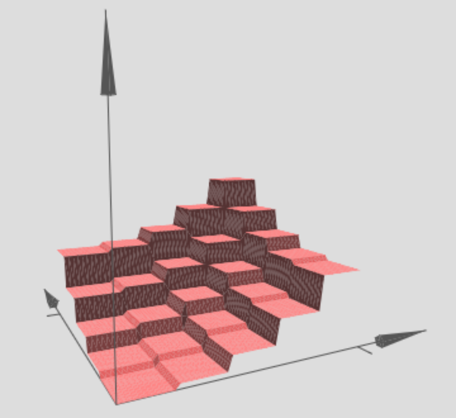

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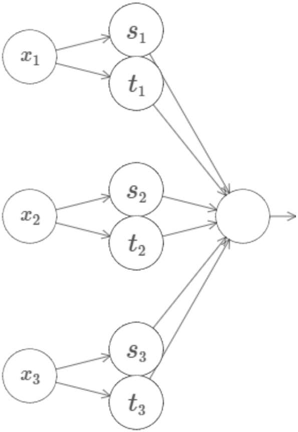
# Universality with multiple inputs

- Three input variables

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# Universality for vector-valued functions

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- $f(x) : \mathcal{R}^m \rightarrow \mathcal{R}^n$

# Universality for vector-valued functions



- $f(x) : \mathcal{R}^m \rightarrow \mathcal{R}^n$
- Can be regarded as $n$ separate real-valued functions $f^1(x_1, \ldots, x_m), \ldots, f^n(x_1, \ldots, x_m)$

# Universality for vector-valued functions

- $f(x) : \mathcal{R}^m \rightarrow \mathcal{R}^n$
- Can be regarded as $n$ separate real-valued functions $f^1(x_1, \ldots, x_m), \ldots, f^n(x_1, \ldots, x_m)$
- Create a network approximating each function $f^i$ and put them all together

# Universal Approximation: Original form



**Theorem 0.1** (UAT, [Cyb89, Hor91]). *Let $\sigma : \mathbb{R} \to \mathbb{R}$ be a non-constant, bounded, and continuous function. Let $I_m$ denote the $m$-dimensional unit hypercube $[0,1]^m$. The space of real-valued continuous functions on $I_m$ is denoted by $C(I_m)$. Then, given any $\varepsilon > 0$ and any function $f \in C(I_m)$, there exist an integer $N$, real constants $v_i, b_i \in \mathbb{R}$ and real vectors $w_i \in \mathbb{R}^m$ for $i = 1, \ldots, N$, such that we may define:*

$$F(\boldsymbol{x}) = \sum_{i=1}^{N} v_i \sigma \left( \boldsymbol{w}_i^T \boldsymbol{x} + b_i \right) = \boldsymbol{v}^\top \sigma \left( \boldsymbol{W}^\top \boldsymbol{x} + \boldsymbol{b} \right)$$

*as an approximate realization of the function $f$; that is,*

$$|F(\boldsymbol{x}) - f(\boldsymbol{x})| < \varepsilon$$

*for all $\boldsymbol{x} \in I_m$.*

# Universal Approximation: Later

- Target function may lie on a space other than the hypercube (has to be bounded)

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- Discontinuous targets can be approximated arbitrarily well
- $\sigma$ can be as general as any nonpolynomial function ($\sigma(z)$ well-defined and different $z \to \infty$ and $z \to -\infty$; at least one side bounded)

# Universal Approximation

- Note that our visual proof had a network with two hidden layers

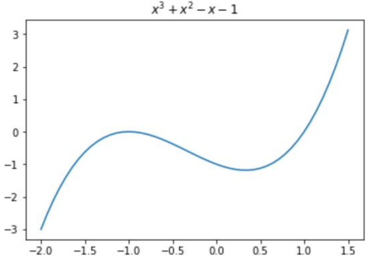
# Universal Approximation



- Note that our visual proof had a network with two hidden layers
- One can show that a single hidden layer can do this

# Universal Approximation using ReLU functions

1. Let's approximate the following function using a bunch of ReLUs:




Example credits: Brendan Fortuner, and https://towardsdatascience.com/

# Universal Approximation using ReLU functions

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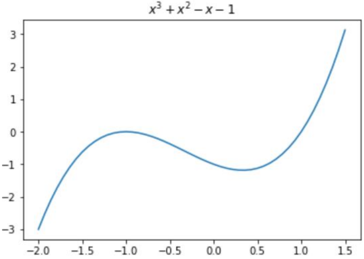
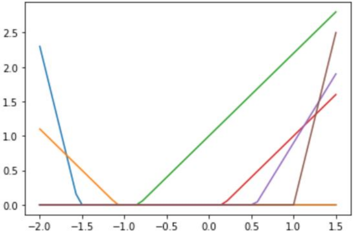
1. $n_1 = ReLU(-5x - 7.7), n_2 = ReLU(-1.2x - 1.3), n_3 = ReLU(1.2x + 1), n_4 = ReLU(1.2x - 0.2), n_5 = ReLU(2x - 1.1), n_6 = ReLU(5x - 5)$

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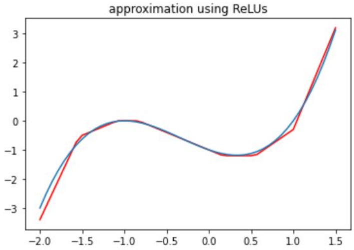
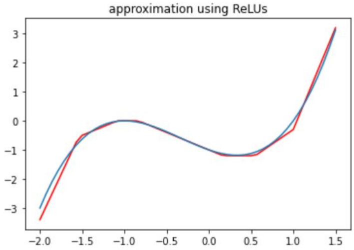
# Universal Approximation using ReLU functions

1. Appropriate combination of these ReLUs:
$-n_1 - n_2 - n_3 + n_4 + n_5 + n_6$



# Universal Approximation using ReLU functions

1. Appropriate combination of these ReLUs:
   $-n_1 - n_2 - n_3 + n_4 + n_5 + n_6$
2. Note that this also holds in case of other activation functions with mild assumptions.



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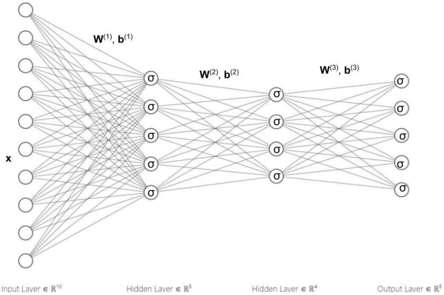
# If one hidden layer is good enough, why Deep learning?

1. May require an infeasible size for the hidden layer
2. May not generalize well
3. Doesn't enable the hierarchical learning

# MLP for regression

1. Output is a continuous variable in $\mathcal{R}^D$
   - Output layer has that many neurons (When $D = 1$, regresses a scalar value)
   - May employ a squared error loss



# MLP for regression

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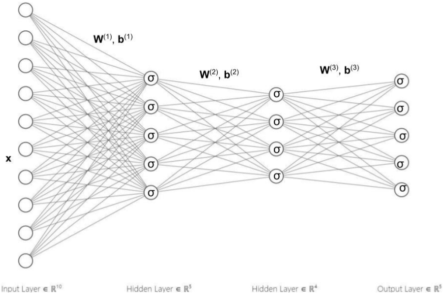
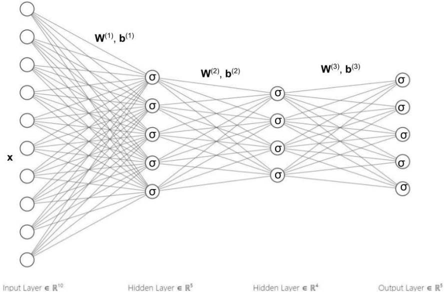
2. Can have an arbitrary depth (number of layers)



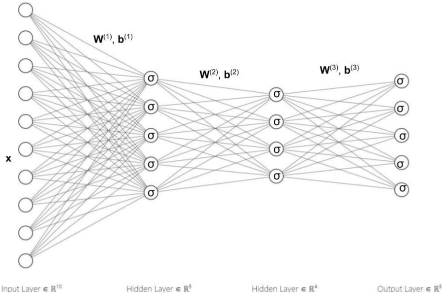
# MLP for classification

1. Categorical output in $\mathcal{R}^C$ where $C$ is the number of categories



# MLP for classification

1. Categorical output in $\mathcal{R}^C$ where $C$ is the number of categories
2. Predicts the scores/confidences/probabilities towards each category
   - Then converts into a pmf
   - Employs loss that compares the probability distributions (e.g. cross-entropy)



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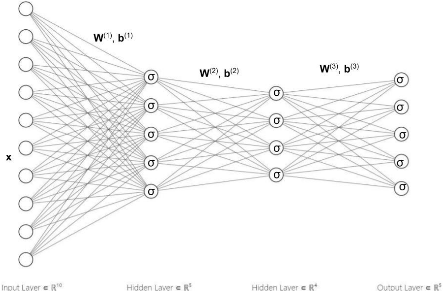

# Extending Linear Classifier

1. Single class: $f(\mathbf{x}) = \sigma(\mathbf{w}^T\mathbf{x} + b)$ from $\mathcal{R}^D \rightarrow \mathcal{R}$ where $\mathbf{w}$ and $\mathbf{x} \in \mathcal{R}^D$

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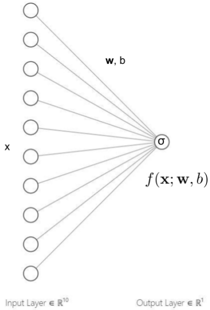
2. Multi-class: $f(\mathbf{x}) = \sigma(\mathbf{W}\mathbf{x} + \mathbf{b})$ from $\mathcal{R}^D \rightarrow \mathcal{R}^C$ where $\mathbf{W} \in \mathcal{R}^{C \times D}$ and $\mathbf{b} \in \mathcal{R}^C$

# Single unit to a layer of Perceptrons





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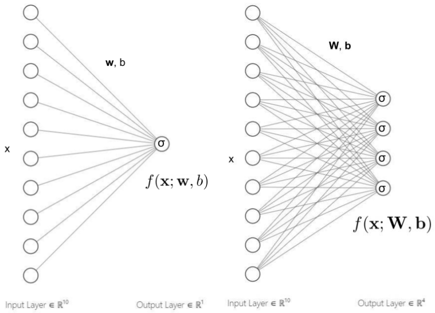

# Single unit to a layer of Perceptrons

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